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Croissance de groupes agissant sur des arbres

Thèse

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Résumé

Au cours de mes cinq années d'assistanat à l'Université de Genève, j'ai eu l'occasion de m'intéresser à de nombreuses questions gravitant autour de la théorie des groupes, et en particulier j'ai eu la très grande chance de collaborer avec Slava Grigorchuk, grâce à ses fréquents passages à Genève. Nous avons ainsi écrit trois articles communs : [**BG00a, BG99a, BG99b**]. Récemment, j'ai aussi entrepris des recherches en commun avec Zoran Šunić, exposées dans l'article [**BŠ00**]. De plus, j'ai écrit indépendamment deux articles sur la croissance du groupe de Grigorchuk : [**Bar98, Bar00d**]. L'exposition de ces résultats représente la première partie de ce travail.

Dans la deuxième partie, je résume mon article [**Bar99**]. Il concerne un sujet aussi étudié par Slava Grigorchuk. Finalement, la troisième partie contient des résultats rédigés ici pour la première fois sur la croissance de certains groupes agissant sur des arbres enracinés.

Première partie

Je rappelle d'abord de la façon la plus complète possible les concepts entourant les groupes agissant sur des arbres enracinés. Ils font l'objet d'une partie de livre écrite par Rostislav Grigorchuk [**Gri00**]. La notion-clé est celle de *groupe à branches*, développée au chapitre 2. De nombreux exemples de groupes [**GS83b**] sont en fait à branches, bien que ce fait n'ait été reconnu que plus tard.

Les deux chapitres suivants décrivent les articles [**BG99a, BG99b**]. Dans ces articles, on appelle *sous-groupe parabolique* le stabilisateur d'un point sur le bord de l'arbre, et on décrit pour cinq exemples de groupes fractals le spectre de l'opérateur de type Hecke pour la représentation quasi-régulière associée à un tel sous-groupe.

Théorème A (voir le théorème 5.2). *Soient $\mathfrak{G}, \tilde{\mathfrak{G}}, \Gamma, \overline{\Gamma}$ et $\overline{\overline{\Gamma}}$ les groupes définis à la section 2.1, avec leurs systèmes de générateurs respectifs. Le spectre de l'opérateur de type Hecke $H = \sum_{\text{générateurs } g} \rho(g)$ associé à la représentation quasi-régulière ρ associée à un sous-groupe parabolique est :*

- *pour $\mathfrak{G}$, une union d'intervalles $[-2, 0] \cup [2, 4]$;*
- *pour $\tilde{\mathfrak{G}}$, l'intervalle $[0, 4]$;*
- *pour Γ et $\overline{\overline{\Gamma}}$, la clôture de l'ensemble*

$$\left\{4, 1, 1 \pm \sqrt{6}, 1 \pm \sqrt{6 \pm \sqrt{6}}, 1 \pm \sqrt{6 \pm \sqrt{6 \pm \sqrt{6}}}, \dots\right\},$$

  qui est l'union d'un ensemble de Cantor de mesure de Lebesgue nulle et d'un ensemble dénombrable ;
- *pour $\overline{\Gamma}$, la clôture de l'ensemble*

$$\left\{4, -2, 1, 1 \pm \sqrt{\frac{9 \pm 3}{2}}, 1 \pm \sqrt{\frac{9 \pm \sqrt{45 \pm 4 \cdot 3}}{2}}, 1 \pm \sqrt{\frac{9 \pm \sqrt{45 \pm 4\sqrt{45 \pm 4 \cdot 3}}}{2}}, \dots\right\},$$

  qui est un ensemble de Cantor de mesure nulle.

En considérant les graphes de Schreier associés à ces groupes et leur sous-groupe, on obtient :

COROLLAIRE B. *Il existe des graphes* 4*-réguliers dont les spectres sont les ensembles décrits dans le théorème A.*

En plus de l'étude de leur spectre, on obtient des résultats sur la décomposition des représentations considérées : On note P le stabilisateur d'un point du bord de l'arbre, et P_n le stabilisateur d'un point à distance n de la racine de l'arbre. Alors la représentation sur $\ell^2(G/P)$ est irréductible, et on détermine complètement la décomposition en irréductibles des représentations sur $\ell^2(G/P_n)$: elle est donnée par les doubles classes de P_n dans G.

THÉORÈME C (voir la section 9 de [**BG99a**]). *Les paires formées d'un des* 5 *groupes G du théorème A et d'un sous-groupe parabolique P sont les paires de Gel'fand. En d'autres termes, les algèbres de Hecke $\mathcal{H}(G,P)$ sont abéliennes.*

Il en va de même pour G et le stabilisateur P_n d'un sommet de niveau n. Par conséquent, il y a une bijection entre les composantes irréductibles de $\ell^2(G/P_n)$ et les doubles classes de P_n dans G, telle que la dimension d'une composante irréductible égale la cardinalité de la P_n-orbite dans G/P_n associée.

Dans le chapitre 6, je décris le premier exemple $\mathfrak{G}$ de Grigorchuk [**Gri80a**], en insistant particulièrement sur quelques résultats nouveaux obtenus par Rostislav Grigorchuk ou par moi. En particulier, nous avons calculé dans [**BG00a**] la structure de plusieurs algèbres de Lie associées à $\mathfrak{G}$. Le résultat principal est la description des constantes de structure de ces algèbres sous la forme d'un graphe, que nous appelons le *graphe de Cayley* de l'algèbre de Lie graduée $L(\mathfrak{G}) = \bigoplus_{n\geq 1} L_n$ sur le corps à deux éléments $\mathbb{F}_2$. Il est formé de sommets à l'abscisse n représentant une base de L_n, avec des arêtes de l'abscisse n à l'abscisse $n+1$ étiquetées par les générateurs de $\mathfrak{G}$ (et donc de $L(\mathfrak{G})$) représentant les constantes de structure de l'algèbre dans la base fixée.

THÉORÈME D (voir le théorème 6.16). *Le graphe de Cayley de $L(\mathfrak{G})$ est comme suit :*

$$\begin{tikzcd}
b \arrow[dr, "a"] & & z & & & & & & & \\
& x \arrow[ur, "{a,b,c}"] \arrow[dr, "{c,d}"] & & & z_1^0 \arrow[r, "a"] & z_1^1 & & & z_2^0 \arrow[r, dashed] & {} \\
a \arrow[ur, "{b,c}"] \arrow[dr, "{c,d}"'] & & x_1^0 \arrow[r, "a"] & x_1^1 \arrow[ur, "{c,d}"] \arrow[dr, "{b,c}"'] & & & x_2^2 \arrow[r, "a"'] & x_2^3 \arrow[ur, "{b,c}"] \arrow[dr, "{b,d}"'] & & \\
& {[a,d]} \arrow[ur, "{b,c}"'] & & & x_2^0 \arrow[r, "a"'] & x_2^1 \arrow[ur, "{b,c}"'] & & & x_3^0 \arrow[r, dashed] & {} \\
d \arrow[ur, "a"'] & & & & & & & & &
\end{tikzcd}$$

où les éléments x_m^r et z_m^r sont définis à la section 6.4.

Le graphe de Cayley de $L(\mathfrak{G})$ du théorème D fait par exemple apparaître les propriétés suivantes de $L(\mathfrak{G}) = \bigoplus_{n\geq 1} L_n$:

- L'espace L_1 est de dimension 3, et $\{b, a, d\}$ en est une base.
- L'espace L_2 est de dimension 2, avec base $\{x, [a, d]\}$. Si on note $A_g(h) = [g, h]$ l'opérateur adjoint, on a

$$\begin{array}{llll}
A_a: & b \mapsto x, & a \mapsto 0, & d \mapsto [a,d] \\
A_b: & b \mapsto 0, & a \mapsto x, & d \mapsto 0 \\
A_c: & b \mapsto 0, & a \mapsto x + [a,d], & d \mapsto 0 \\
A_d: & b \mapsto 0, & a \mapsto [a,d], & d \mapsto 0
\end{array}$$

– Les espaces L_{2^m+1+r} sont de dimension 2 (engendrés par $\{x_m^r, z_{m-1}^r\}$) quand $0 \leq r < 2^{m-1}$, et de dimension 1 (engendrés par x_m^r) quand $2^{m-1} \leq r < 2^m$.
– Les seuls opérateurs $A_g : L_{2^m+1+r} \to L_{2^m+1+r+1}$, pour $g \in \{a, b, c, d\}$, satisfaisant $A_g(x_m^r) = x_m^{r+1}$ sont ceux correspondant à la r-ième lettre de $\sigma^m(a)$, pour la substitution σ de la section 6.2.

En résumé, on montre que $\mathfrak{G}$ est de largeur finie. On réfute ainsi la conjecture 6.10, due à Efim Zel'manov, en exhibant un nouveau type de pro-2-groupe de largeur finie : la pro-2-complétion de $\mathfrak{G}$.

Dans la dernière section, je rappelle les résultats de mes deux articles [**Bar98, Bar00d**]. Ils donnent les bornes suivantes sur la fonction de croissance

$$\gamma(n) = \#\{g \in G | \, g = s_1 \dots s_n, s_i \in \{1, a, b, c, d\}\}$$

du groupe $\mathfrak{G}$:

THÉORÈME E (voir les théorèmes 6.17 et 6.19). *La fonction de croissance $\gamma(n)$ du groupe de Grigorchuk $\mathfrak{G}$ satisfait*

$$e^{n^\alpha} \precsim \gamma(n) \precsim e^{n^\beta},$$

avec $\alpha = 0.5157$ et $\beta = \log(2)/\log(2/\eta) \approx 0.767$, où $\eta \approx 0.811$ est la racine réelle du polynôme $X^3 + X^2 + X - 2$.

Dans le chapitre 7, je décris une généralisation de nombreux résultats de croissance des mots et des périodes à une grande famille, définie algébriquement, de groupes agissant sur des arbres. Ces résultats ont été obtenus avec Zoran Šunić [**BŠ00**]. À titre d'exemple, on montre :

THÉORÈME F (voir le théorème 7.5). *Si ω est une suite r-homogène (voir la définition 7.4), alors la fonction de croissance du groupe G_ω (définie en 7.1) agissant sur l'arbre d-régulier satisfait*

$$e^{n^\alpha} \precsim \gamma_\omega(n) \precsim e^{n^\beta}$$

avec $\alpha = \frac{\log(d)}{\log(d)-\log\frac{1}{2}}$ et $\beta = \frac{\log(d)}{\log(d)-\log(\eta_r)}$, où η_r est la racine réelle du polynôme $X^r + X^{r-1} + X^{r-2} - 2$.

Si de plus G_ω agit régulièrement sur le premier niveau de l'arbre, alors la fonction de croissance des périodes

$$\pi_\omega(n) = \max\{\text{ordre}(g) | \, g = s_1 \dots s_n, s_i \in \{1, a, b, c, d\}\}$$

satisfait

$$\pi_\omega(n) \precsim n^{\log_{1/\eta_r}(d)}.$$

Deuxième partie

La deuxième partie décrit mon premier article, qui montre une généralisation de la formule de Grigorchuk reliant la croissance au rayon spectral. Étant donné un graphe $\mathfrak{G}$ avec deux sommets $\star, \dagger$ fixés, on définit la série de croissance des chemins $G(t) = \sum_{\gamma \in [\star, \dagger]} t^{|\gamma|}$, où la somme parcourt les chemins de $\star$ à $\dagger$, et la série $F(u,t) = \sum_{\gamma \in [\star, \dagger]} t^{|\gamma|} u^{\text{nb}(\gamma)}$, où $\text{nb}(\gamma)$ est le nombre de points où γ rebrousse son chemin. Le résultat principal est :

THÉORÈME G (voir le théorème 8.5). *Si $\mathfrak{G}$ est d-régulier, on a*

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}. \tag{1}$$

En particulier, $F(1,t) = G(t)$ et $F(0,t)$ compte les chemins sans points de rebroussement. Si $\mathfrak{G}$ est un arbre et $\star = \dagger$, on a $F(0,t) = 1$ car un arbre ne contient aucun circuit élémentaire non-trivial, et (1) donne aisément la série comptant les circuits dans un arbre d-régulier

$$G(t) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}},$$

déjà obtenue par Harry Kesten [**Kes59**].

L'équation (1) donne aussi une relation entre les rayons de convergence de $F(0,t)$ et $G(t)$:

COROLLAIRE H. *Si on note $1/\alpha$ le rayon de convergence de $F(0,t)$ et $1/(d\nu)$ le rayon de convergence de $G(t)$, on a*

$$\nu = \begin{cases} \frac{\sqrt{d-1}}{d}\left(\frac{\alpha}{\sqrt{d-1}} + \frac{\sqrt{d-1}}{\alpha}\right) & \textit{si } \alpha > \sqrt{d-1}, \\ \frac{2\sqrt{d-1}}{d} & \textit{sinon.} \end{cases}$$

Cette équation, appelée la ń formule de Grigorchuk ż, constitue le résultat principal de [**Gri80b**]. Si $\mathfrak{G}$ est le graphe de Cayley d'un groupe Γ relativement à un système de générateurs S de cardinalité d, on appelle généralement α la *cocroissance* de Γ et ν le *rayon spectral* de la marche aléatoire simple sur $\mathfrak{G}$. La série G s'appelle aussi la *série de Green* de $\mathfrak{G}$. Elle contient de nombreuses informations sur Γ ; par exemple, le théorème de Kesten affirme que Γ est moyennable si et seulement si $\nu = 1$.

Je considère ensuite les produits libres de graphes, qui sont un analogue naturel des produits libres de groupes, et j'obtiens une formule reliant les fonctions G du produit aux fonctions G des facteurs. Rappelons qu'un graphe $\mathfrak{G}$ est *transitif sur ses sommets* si son groupe d'automorphismes agit transitivement sur les sommets de $\mathfrak{G}$.

THÉORÈME I (voir le théorème 11.2). *Soient $G_{\mathfrak{E}}$ et $G_{\mathfrak{F}}$ les séries de Green de deux graphes $\mathfrak{E}$ et $\mathfrak{F}$ transitifs sur leurs sommets. Alors*

$$\frac{1}{(tG_{\mathfrak{E}*\mathfrak{F}})^{-1}} = \frac{1}{(tG_{\mathfrak{E}})^{-1}} + \frac{1}{(tG_{\mathfrak{F}})^{-1}} - \frac{1}{t},$$

où $F^{-1}(t)$ est l'inverse formel de la série F, c'est-à-dire une série G telle que $G(F(t)) = F(G(t)) = t$.

Une équation équivalente à celle-ci, mais de manière non-triviale, apparaît dans un article de Gregory Quenell [**Que94**], et, dans un langage complètement différent (celui des variables aléatoires non-commutatives), dans un article de Dan Voiculescu [**Voi90**, Theorem 4.5].

Troisième partie

Finalement, dans la troisième partie, je donne des résultats jamais publiés concernant la croissance de groupes *GGS*. Ce sont des groupes définis ainsi : on fixe un entier d, un sous-groupe A du groupe symétrique $\mathfrak{S}_d$ sur d lettres et une suite $\epsilon = (\epsilon_1, \dots, \epsilon_{d-1})$ d'éléments de A. Le groupe A agit sur les suites $\{1, 2, \dots, d\}^*$ en n'agissant que sur la première lettre, et un automorphisme t est défini par

$$t(d \dots d i_0 i_1 \dots i_n) = d \dots d i_0 \epsilon_{i_0}(i_1) i_2 \dots i_n,$$

où $i_0 \neq d$ (et t fixe la suite $d \dots d \dots d$). Un groupe *GGS* est un groupe engendré par de tels A et t ; la terminologie fait référence à Rostislav Grigorchuk, Narain Gupta et Saïd Sidki [**Bau93**].

Le résultat principal est :

THÉORÈME J (voir le théorème 14.2). *Soit G un groupe GGS. Alors soit G est virtuellement abélien, soit G est de croissance intermédiaire.*

Ce résultat s'applique en particulier au groupe suivant : on prend $d = 4$, $A = \langle a \rangle$ où a est le 4-cycle $(1, 2, 3, 4)$, et $\epsilon = (a, 1, a^3)$. Je réponds ainsi à une question que Rostislav Grigorchuk a posée il y a dix ans, en montrant que ce groupe est de croissance intermédiaire.

Il est aussi valable pour les trois exemples $\Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$ de la section 2.1. En particulier, il étend le résultat de [**FG85, FG91**] affirmant la croissance sous-exponentielle de Γ.

J'indique aussi des critères déterminant quand un groupe *GGS* est infini, juste-infini, de torsion, etc.

Annexes

Finalement je liste tous les articles à la rédaction desquels j'ai participé, en j'en inclus ceux qui ont le plus grand rapport avec les sujets traités dans cette thèse. Il s'agit de [**Bar98, Bar00d**] que j'ai écrits seul, [**BC99**] écrit avec Tullio Ceccherini-Silberstein, et [**BG00a, BG99b, BG99a**] écrits avec Rostislav Grigorchuk. En revanche, j'omets [**Bar99**] auquel la deuxième partie est consacrée.

Première partie

Introduction aux groupes enracinés

CHAPITRE 1

Arbres

Dans son premier article [**Gri80a**], Rostislav Grigorchuk introduit son premier groupe comme un sous-groupe du groupe des transformations linéaires par morceaux de l'intervalle. Depuis, il a semblé plus naturel de le définir comme un sous-groupe du groupe des automorphismes d'un arbre enraciné.

Ce passage s'effectue en plusieurs étapes : on fixe un entier d, et on note $B = \{0, 1, \dots, d-1\}$. L'écriture en base d donne une bijection τ entre $[0, 1[$ et les suites $(x_0, x_1, x_2, \dots) \in \Sigma^{\mathbb{N}}$ qui ne se terminent pas par $\overline{d-1}$.

L'ensemble $B^{\mathbb{N}}$ est un espace topologique (pour la topologie de Tychonoff), et même un espace métrique compact, pour la distance

$$d(x, y) = d^{-\min\{n \mid x_n \neq y_n\}}$$

(où par convention $\min \emptyset = \infty$), et donc un espace mesuré, pour la mesure de Bernouilli définie sur les ouverts cylindriques $\sigma B^{\mathbb{N}}$ (avec $\sigma \in B^*$) par

$$\mu(\sigma B^*) = d^{-|\sigma|}.$$

On peut maintenant considérer les actions sur $B^{\mathbb{N}}$ au lieu de $\tau([0, 1[)$, en considérant les actions à en ensemble de mesure nulle près.

Dans une deuxième étape, on suppose que le groupe agit par homéomorphismes linéaires par morceaux de $[0, 1[$, dont les pentes sont de la forme d^i avec $i \in \mathbb{Z}$ et dont les points de discontinuité de la pente sont de la forme id^j avec $i, j \in \mathbb{Z}$. Il suit alors que l'action, transposée dans le domaine $B^{\mathbb{N}}$, préserve les ouverts cylindriques $\sigma B^{\mathbb{N}}$, c'est-à-dire est de la forme $\sigma B^{\mathbb{N}} \mapsto \tau B^{\mathbb{N}}$. On obtient ainsi une action du groupe sur B^*, donnée par $\sigma \mapsto \tau$. Or ce dernier ensemble B^* a la structure d'un arbre, comme on va le voir.

Ce passage d'une action sur $[0, 1[$ à une action sur un arbre enraciné a été déjà utilisé, par exemple, dans l'étude du groupe de Thompson [**CFP96, Röv99b**].

Definition 1.1. Un *digraphe* est un couple d'ensembles $\mathfrak{G} = (S, A)$, muni de deux applications $\alpha, \omega : A \to S$. Les éléments de S sont appelés les *sommets* de $\mathfrak{G}$, et ceux de A ses *arêtes*. $\mathfrak{G}$ est dit *fini* si S et A sont des ensembles finis. $\alpha(a)$ et $\omega(a)$ sont les *extrémités* de l'arête a.

La définition ci-dessus sera constamment utilisée dans ce travail. Elle se distingue de la définition habituelle en combinatoire en ce que les arêtes sont toujours supposées orientées, elles peuvent être des boucles, et il peut y avoir plusieurs arêtes entre deux sommets.

Definition 1.2. Soit Σ un alphabet fini. On désigne par Σ^* le monoïde libre sur Σ, c'est-à-dire l'ensemble des mots finis formés de lettres de Σ, avec pour opération la concaténation.

Soit M un monoïde, engendré par un ensemble Σ. Le *de Cayley* de M relativement à Σ est le digraphe étiqueté $\mathcal{C}(M, \Sigma)$ dont l'ensemble de sommets est M, et dont les arêtes sont les (m, ms), étiquetées par s, pour tous les $m \in M$ et $s \in \Sigma$.

Soit Σ un alphabet fini. On appelle *arbre régulier enraciné* le graphe de Cayley du monoïde libre sur Σ, et on le note $\mathcal{T}_\Sigma$, ou $\mathcal{T}$ s'il n'y a pas d'ambiguïté sur Σ.

Il y a une correspondance naturelle entre les sommets de $\mathcal{T}$ et les chemins géodésiques (c'est-à-dire sans aller-retour) finis issus de $\emptyset$ dans $\mathcal{T}$.

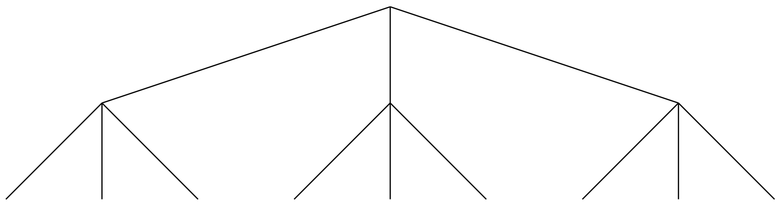


FIG. 1.1 – La ń partie supérieure ż de $\mathcal{T}_{\{1,2,3\}}$

DEFINITION 1.3. Soit $\mathcal{T}_\Sigma$ un arbre. Le *bord* de $\mathcal{T}_\Sigma$ est l'ensemble $\partial\mathcal{T}_\Sigma = \Sigma^{\mathbb{N}}$, muni de la topologie de Tychonoff. On l'identifie à l'espace des chemins géodésiques infinis issus de $\emptyset$ dans $\mathcal{T}_\Sigma$.

Si on prend $\Sigma = \{0, 1, \ldots, d-1\}$ comme plus haut, on a alors l'arbre $\mathcal{T}_\Sigma = \Sigma^*$, dont le bord est $\partial\mathcal{T}_\Sigma = \Sigma^{\mathbb{N}}$, équivalent à un ensemble de mesure nulle près à $[0, 1]$.

Bien que ce ne soit apparu que plus tard, le fait de voir les groupes de Grigorchuk comme des groupes d'automorphismes d'un arbre régulier enraciné est très fructueux : on fait apparaître ainsi de nombreuses propriétés géométriquement claires mais algébriquement difficiles à formuler. On pense par exemple à la notion de *profondeur* ou de *portrait* d'un élément — voir la définition 1.6.

Des traitements algébriques des groupes agissant sur des arbres ont été tentés : on peut se débarrasser entièrement de la notion d'arbre pour ne considérer que des groupes G construits inductivement à partir de produits en couronne $G' \wr A$ où G' est un groupe du même type et A est un groupe fini. Cette approche a été suivie par Lev Kaloujnine [**Kal45, Kal48**], son élève Vitaliĭ Sushchanskiĭ [**Suš79**], John Wilson [**GW99**], Zoran Šunik et l'auteur [**BŠ00**] (voir le chapitre 7), et bien d'autres encore.

Notons que Σ^* est infini dès que Σ est non-vide ; la ń partie supérieure ż de Σ^* est représentée dans la figure 1.1, où $|\Sigma| = 3$. Il faut aussi remarquer que dans cette théorie les arbres poussent traditionnellement vers le bas, et que la racine de l'arbre est représentée en haut.

En parlant d'un arbre enraciné, les conventions suivantes seront respectées : il sera dit *d-régulier*, où $d = |\Sigma|$, et ceci bien qu'en chaque sommet excepté la racine il y ait $d+1$ arêtes, d sortantes et une entrante. On supposera toujours

$$\Sigma = \mathbb{Z}/d\mathbb{Z} = \{1, 2, \ldots, d\}$$

avec ce choix de représentants, qui rendent les notations plus simples. On identifiera les sommets à des suites finies de symboles pris dans Σ, et la racine à la suite vide $\emptyset$.

Soit σ un sommet de $\mathcal{T}_\Sigma$. Le Σ^*-module à droite $\sigma\Sigma^*$ des mots commençant par σ, est appelé le *sous-arbre en* σ et est noté $(\mathcal{T}_\Sigma)_\sigma$. On a les décompositions suivantes :

$$\mathcal{T}_\Sigma = \{\emptyset\} \cup \bigcup_{s\in\Sigma} (\mathcal{T}_\Sigma)_s,$$
$$\partial\mathcal{T}_\Sigma = \bigcup_{s\in\Sigma} \partial(\mathcal{T}_\Sigma)_s.$$

1.1. Automorphismes

On note $\mathsf{Aut}\,\mathcal{T}$ le groupe des automorphismes de $\mathcal{T}$. Ce sont les bijections de $\mathcal{T}$ qui préservent la structure de digraphe :

$$\mathsf{Aut}\,\mathcal{T} = \{g : \mathcal{T} \xrightarrow{1-1} \mathcal{T} \mid (\sigma, \tau) \in A \Leftrightarrow (g\sigma, g\tau) \in A\},$$

où A désigne l'ensemble des arêtes de $\mathcal{T}$.

LEMMA 1.4. *Soit $g \in \mathsf{Aut}\,\mathcal{T}$. Alors g préserve les préfixes et la distance à la racine :*

$$|\sigma| = |g\sigma|, \qquad g(\sigma\tau) \in g(\sigma)\Sigma^*.$$

DÉMONSTRATION. g, étant une bijection du graphe, doit envoyer l'unique chemin géodésique de $\emptyset$ à σ sur l'unique chemin géodésique de $\emptyset$ à $g\sigma$, et ces chemins ont nécessairement même longueur. Un chemin de $\emptyset$ à $g(\sigma\tau)$ doit nécessairement passer par $g(\sigma)$. □

Il s'ensuit que tout $g \in \mathsf{Aut}\,\mathcal{T}$ agit par permutation sur Σ^n, l'ensemble des branches à profondeur n. En particulier, g agit sur Σ et on a une surjection $\mathsf{Aut}\,\mathcal{T} \twoheadrightarrow \mathfrak{S}_\Sigma$, où $\mathfrak{S}_\Sigma = \mathfrak{S}_d$ désigne le groupe symétrique sur d lettres.

Plus généralement, soit $\mathsf{Stab}(\sigma)$ le stabilisateur dans $\mathsf{Aut}\,\mathcal{T}$ du sommet σ. On forme

$$\mathsf{Stab}(n) = \bigcap_{\sigma \in \Sigma^n} \mathsf{Stab}(\sigma).$$

Soit π_n la surjection $\mathsf{Aut}\,\mathcal{T} \twoheadrightarrow \mathfrak{S}_{\Sigma^n}$. On a alors $\mathsf{Stab}(n) = \mathrm{Ker}(\pi_n)$, et π_n admet une section dans $\mathsf{Aut}\,\mathcal{T}$, donnée par l'action sur les premiers n symboles de tout mot. On a donc toujours $\mathfrak{S}_{\Sigma^n} \subset \mathsf{Aut}\,\mathcal{T}$ canoniquement ainsi : si $\Pi \in \mathfrak{S}_{\Sigma^n}$,

$$\Pi(\sigma_1 \dots \sigma_n \tau) = \Pi(\sigma_1 \dots \sigma_n)\tau, \tag{2}$$

et l'action de Π sur des mots de longueur inférieure à n est imposée par le lemme 1.4.

Soit σ un sommet de $\mathcal{T}$. On a une injection $\phi_\sigma : \mathsf{Aut}\,\mathcal{T} \to \mathsf{Stab}(\sigma) \subset \mathsf{Aut}\,\mathcal{T}$ induite par l'injection de $\mathcal{T}_\sigma$ dans $\mathcal{T}$. En formules, il s'agit de

$$\phi_\sigma(g) : \begin{cases} \sigma\tau & \mapsto \sigma g(\tau) \\ \tau \notin \sigma\Sigma^* & \mapsto \tau. \end{cases} \tag{3}$$

Il est clair que tout $g \in \mathsf{Stab}(n)$ induit par restriction des $g_\sigma \in \mathsf{Stab}(\sigma)$ pour tout $\sigma \in \Sigma^n$, et qu'une collection $(g_\sigma)_{\sigma \in \Sigma^n}$ quelconque d'éléments de $\mathsf{Aut}\,\mathcal{T}$ produit, par assemblage via $(\phi_\sigma)_{\sigma \in \Sigma^n}$, un élément de $\mathsf{Stab}(n)$. On a donc le résultat suivant, dû à Lev Kaloujnine :

PROPOSITION 1.5.

$$\mathsf{Aut}\,\mathcal{T} = (\mathsf{Aut}\,\mathcal{T} \times \cdots \times \mathsf{Aut}\,\mathcal{T}) \rtimes \mathfrak{S}_\Sigma,$$

où le produit direct de d copies de $\mathsf{Aut}\,\mathcal{T}$ s'injecte dans $\mathsf{Aut}\,\mathcal{T}$ via $\phi_1 \times \cdots \times \phi_d$, et $\mathfrak{S}_\Sigma$ s'injecte naturellement dans $\mathsf{Aut}\,\mathcal{T}$ comme en (2).

Dans ce qui suit, on aura toujours les notations suivantes :

$$\phi_\sigma \ : \mathsf{Aut}\,\mathcal{T} \to \mathsf{Stab}(\sigma); \tag{4}$$

son inverse,

$$\psi_\sigma \ : \mathsf{Stab}(\sigma) \to \mathsf{Aut}\,\mathcal{T}; \tag{5}$$

leur assemblage,

$$\psi_n \ : \begin{cases} \mathsf{Stab}(n) & \to (\mathsf{Aut}\,\mathcal{T})^{\Sigma^n} \\ g & \mapsto (\psi_{1\dots1}(g), \dots, \psi_{d\dots d}(g); \end{cases} \tag{6}$$

et l'application quotient

$$\pi_n \ : \mathsf{Aut}\,\mathcal{T} \to \mathfrak{S}_{\Sigma^n}. \tag{7}$$

On notera aussi $\psi = \psi_1$ et $\pi = \pi_1$.

Tout élément $g \in \mathsf{Aut}\,\mathcal{T}$ peut être identifié à son *portrait*, défini comme suit par induction :

Definition 1.6. Soit $G < \mathsf{Aut}\,\mathcal{T}$ un groupe et $S \subset G$ un sous-ensemble. Le *S-portrait* de $g \in G$ par rapport à S est un sous-arbre de $\mathcal{T}$, dont les sommets intérieurs sont étiquetés par $\mathfrak{S}_\Sigma$ et dont les sommets terminaux sont étiquetés par S. Il est défini inductivement comme suit : si $g \in S$, son portrait est le sous-arbre de $\mathcal{T}$ à un sommet (la racine), dont l'étiquette est g lui-même. Sinon, l'étiquette de la racine est $\pi_1(g)$, et les étiquetages des sous-arbres $\mathcal{T}_1, \dots, \mathcal{T}_d$ sont les S-portraits de $\psi_1(g\pi_1(g)^{-1}), \dots, \psi_d(g\pi_1(g)^{-1})$.

Si $S = \emptyset$, on parle simplement du *portrait* de g. Le sous-arbre sur lequel il repose est alors nécessairement $\mathcal{T}$ tout entier. Inversement, un élément est dit *S-fini* si l'arbre de son S-portrait est fini.

La *profondeur* de $g \in G$ est la profondeur de l'arbre de son portrait, c'est-à-dire la longueur maximale d'un chemin commençant à la racine et sans rebroussement. On la note $\partial(g) \in \mathbb{N} \cup \{\infty\}$.

Considérons par exemple le groupe $\mathfrak{G}$ défini au chapitre 6. On a $d = 2$, donc on peut représenter les portraits sous la forme $(x_1, x_2)\epsilon$ où x_1 et x_2 sont des portraits et ϵ est soit 1 soit a.

Notons $\Pi(g)$ le portrait de g, et prenons d'abord $S = \{1, a, b, c, d\}$. On a

$$\Pi(1) = 1, \quad \Pi(a) = a, \quad \Pi(b) = b, \quad \Pi(ab) = (c, a)a,$$
$$\Pi(aba) = (c, a), \quad \Pi(abab) = (\Pi(ca), \Pi(ac)) = ((a, d)a, (d, a)a).$$

En revanche, si on prend $S = \{1\}$, on a

$$\Pi(1) = 1, \quad \Pi(a) = a, \quad \Pi(b) = (a, (a, (1, (a, \dots)))),$$
$$\Pi(ab) = ((a, (1, (a, (a, \dots)))), a)a, \quad \Pi(aba) = ((a, (1, (a, (a, \dots)))), a),$$
$$\Pi(abab) = ((a, (1, (a, (a, \dots)))a, ((1, (a, (a, \dots))), a)a).$$

Soit ι un portrait dans $\mathcal{T}$, c'est-à-dire une application $\iota : \mathcal{T} \to S \cup \mathfrak{S}_\Sigma$. On peut construire un automorphisme g_ι de $\mathcal{T}$ comme suit :

$$g_\iota(\sigma_1\sigma_2\dots) = \begin{cases} \iota_\emptyset(\sigma_1\sigma_2\dots) & \text{si } \iota_\emptyset \in S, \\ \iota_\emptyset(\sigma_1)g_{\iota'}(\sigma_2\dots) & \text{sinon, où } \iota'_\tau = \iota_{\sigma_1\tau} \text{ pour tout } \tau \in \mathcal{T}. \end{cases}$$

et il est clair qu'on récupère ainsi l'élément de $\mathsf{Aut}\,\mathcal{T}$ dont on a pris le S-portrait.

Remarquons encore que $\mathsf{Aut}\,\mathcal{T}$ est un groupe profini, égal à la limite inverse des $\mathfrak{S}_{\Sigma^n}$, et qu'il agit par isométries sur l'espace compact $\partial\mathcal{T}$ — voir le chapitre 2.

1.2. Sous-groupes

Soit maintenant G un sous-groupe de $\mathsf{Aut}\,\mathcal{T}$. On définit les groupes suivants, associés à G et calqués sur les définitions de la section précédente :

- pour σ un sommet, le stabilisateur $\mathsf{Stab}_G(\sigma) = \mathsf{Stab}(\sigma) \cap G$;
- de façon similaire, $\mathsf{Stab}_G(n) = \mathsf{Stab}(n) \cap G$;
- le *sur-groupe de sommet G^σ* du sommet σ, qui est l'image de $\mathsf{Stab}_G(\sigma)$ par ψ_σ ;
- le *sous-groupe de sommet G_σ* du sommet σ, qui est l'intersection de G et de $\mathsf{Aut}\,\mathcal{T}_\sigma$; on l'appelle aussi le *stabilisateur restreint* $\mathsf{Rist}_G(\sigma)$. En formules,

$$\mathsf{Rist}_G(\sigma) = \{g \in G \mid g\tau = \tau \text{ pour tout } \sigma \notin \sigma\Sigma^*\}. \tag{8}$$

  On peut donner cette définition alternative de $\mathsf{Rist}_G(\sigma)$: ce sont les éléments de G dont le portrait est trivial en-dehors de $\mathcal{T}_\sigma$;
- de façon similaire, $\mathsf{Rist}_G(n) = \prod_{\sigma\in\Sigma^n} \mathsf{Rist}_G(\sigma)$.

Proposition 1.7. *On a les inclusions suivantes :*

$$\mathsf{Rist}_G(n) \leq \mathsf{Stab}_G(n) \hookrightarrow \prod_{\sigma\in\Sigma^n} G^\sigma.$$

DEFINITION 1.8. Soit G un groupe agissant sur l'arbre enraciné $\mathcal{T}_\Sigma$. Il est dit *sphériquement transitif* si l'action de G sur Σ^n est transitive, pour tout $n \in \mathbb{N}$.

G est dit *fractal* si pour tout sommet σ de $\mathcal{T}_\Sigma$ on a $G^\sigma = G$, c'est-à-dire si $\mathsf{Stab}_G(\sigma)$ est isomorphe à G par l'identification ψ_σ.

G a la *propriété de congruence* si tout sous-groupe d'indice fini de G contient $\mathsf{Stab}_G(n)$ pour un n suffisamment grand.

On constate aisément que G est sphériquement transitif si et seulement si l'action de G sur le bord $\partial\mathcal{T}$ de l'arbre est ergodique.

LEMMA 1.9. *G est fractal si et seulement si l'image de $\psi(G) < (\mathsf{Aut}\,\mathcal{T})^\Sigma$ est un sous-produit direct sur G, c'est-à-dire si $\psi_{|G}$ est une inclusion dans $G \times \cdots \times G$ qui est surjective sur chaque facteur.*

DÉMONSTRATION. Si $p_i\psi_{|G} \neq G$ pour une projection p_i sur le sommet i, alors $\mathsf{Stab}_G(i)_{|i} \neq G$ donc G n'est pas fractal. On suppose maintenant que $\psi_{|G}$ est une inclusion dans un sous-produit direct, et on montre par induction que $\mathsf{Stab}_G(\sigma)_{|\sigma} \cong G$ pour tout $\sigma \in \mathcal{T}_\Sigma$.

La base de l'induction, pour $|\sigma| = 1$, équivaut à l'hypothèse. Maintenant par induction $G \to G^{\Sigma^{n-1}}$ est une inclusion dans un sous-produit direct, et chaque facteur G est appliqué sur G^Σ par $\psi_{|G}$. La composée de deux inclusions dans des sous-produits directs est encore de la même forme, donc $G \to G^{\Sigma^n}$ est bien une inclusion dans un sous-produit direct. $\square$

LEMMA 1.10. *Un groupe fractal agissant sur $\mathcal{T}_\Sigma$ est sphériquement transitif si et seulement s'il agit transitivement sur le premier niveau Σ de l'arbre $\mathcal{T}_\Sigma$.*

DÉMONSTRATION. Soit G fractal, et supposons qu'il est transitif sur Σ. On montre par induction qu'il est transitif sur Σ^n pour tout $n \in \mathbb{N}$.

La base de l'induction équivaut à l'hypothèse. Soit $\sigma \in \Sigma^n$ un sommet. Du fait que G est transitif sur Σ, on a un sommet $1\tau_2 \dots \tau_n \in \Sigma^n$ dans la G-orbite de σ. Si de plus G est transitif sur Σ^{n-1}, alors le groupe de sommet G^1, isomorphe à G, est transitif sur $1\Sigma^{n-1}$, et $1 \dots 1 \in \Sigma^n$ est dans la G-orbite de σ, donc G est transitif sur Σ^n. $\square$

1.3. Groupes profinis

On se rappelle que $\partial\mathcal{T}$, l'ensemble des suites infinies sur Σ, est un espace métrique pour la distance
$$\delta(\sigma,\tau) = \max\{d^{-n} |\, \sigma_n \neq \tau_n\},$$
avec la convention habituelle $\max\emptyset = 0$. Comme tel, $\partial\mathcal{T}$ est homéomorphe à l'ensemble de Cantor. Le groupe $\mathsf{Aut}\,\mathcal{T}$ agit sur $\partial\mathcal{T}$ par isométries, et on a même $\mathsf{Aut}\,\mathcal{T} = \mathsf{Isom}\partial\mathcal{T}$.

On peut maintenant mettre une métrique sur $\mathsf{Aut}\,\mathcal{T}$: pour $f, g \in \mathsf{Aut}\,\mathcal{T}$, on pose
$$\delta(f,g) = \max\{2^{-|\sigma|} |\, f(\sigma) \neq g(\sigma)\}.$$
Cette distance induit une structure d'espace compact séparé sur $\mathsf{Aut}\,\mathcal{T}$, dans laquelle deux éléments sont ń proches ż si leurs actions coïncident sur un grand sous-arbre $\bigcup_{i\le n} \Sigma^i$ de $\mathcal{T}$. Il s'ensuit que $\mathsf{Aut}\,\mathcal{T}$ est un groupe profini. Clairement, la distance δ est invariante à droite et à gauche.

On peut aussi voir la structure profinie de $\mathsf{Aut}\,\mathcal{T}$ en considérant ses quotients finis
$$\mathsf{Aut}(\mathcal{T})_n = \mathsf{Aut}\,\mathcal{T} / \mathsf{Stab}(n) = \mathfrak{S}_\Sigma \wr \cdots \wr \mathfrak{S}_\Sigma$$
donnés par la restriction de l'action à Σ^n, ou de façon équivalente au sous-arbre $\bigcup_{i\le n} \Sigma^i$, et on a
$$\mathsf{Aut}\,\mathcal{T} = \varprojlim_{n\to\infty} \mathsf{Aut}(\mathcal{T})_n.$$
La topologie donnée par ce système projectif coïncide avec celle donnée par la métrique δ.

Les pro-sous-groupes de Sylow de $\mathsf{Aut}\,\mathcal{T}$ peuvent être obtenus ainsi : on fixe un premier p. On note π_{ij}, pour $i \geq j$, la projection naturelle $\mathsf{Aut}(\mathcal{T})_i \twoheadrightarrow \mathsf{Aut}(\mathcal{T})_j$, et π_j la projection $\mathsf{Aut}\,\mathcal{T} \twoheadrightarrow \mathsf{Aut}(\mathcal{T})_j$.

On choisit un sous-groupe de Sylow P_1 de $\mathsf{Aut}(\mathcal{T})_1$. Ensuite, pour tout $n \geq 2$, on choisit un sous-groupe de Sylow P'_n de $\pi^{-1}_{n,n-1}(P_{n-1})$, qui est un p-sous-groupe de $\mathsf{Aut}(\mathcal{T})_n$ qu'on complète en P_n, un p-Sylow de $\mathsf{Aut}(\mathcal{T})_n$. On pose maintenant

$$P = \{g \in \mathsf{Aut}\,\mathcal{T} \,|\, \pi_j(g) \in P_j \text{ pour tout } j \in \mathbb{N}\}.$$

P est un pro-sous-groupe de Sylow de $\mathsf{Aut}\,\mathcal{T}$, et ils peuvent tous être obtenus de cette manière.

Au cas où d est un nombre premier, on peut exhiber aisément un d-Sylow de $\mathsf{Aut}\,\mathcal{T}$, qu'on notera $\mathsf{Aut}_*\,\mathcal{T}$: choisissons un d-cycle de $\mathfrak{S}_d$, par exemple $a = (1, \dots, d)$, et considérons le d-Sylow $\langle a\rangle$ de $\mathfrak{S}_d$. Alors $\mathsf{Aut}_*\,\mathcal{T}$ est formé de tous les $g \in \mathsf{Aut}\,\mathcal{T}$ dont le portrait n'a d'étiquettes que dans $\langle a\rangle$. Il est un pro-d groupe, et est égal à la limite inverse des produits en couronne itérés $\mathbb{Z}/d\mathbb{Z} \wr \cdots \wr \mathbb{Z}/d\mathbb{Z}$.

On connaît mal les propriétés et la structure de $\mathsf{Aut}\,\mathcal{T}$. Il se décompose comme produit en couronne, ce qui a fait l'objet de la thèse de Lev Kaloujnine [**Kal48**]. On sait toutefois ceci : on se rappelle qu'un groupe profini est dit à *base dénombrable* si la topologie du groupe est ensemble dénombrable de voisinages de l'identité ; ou, ce qui est équivalent, s'ils sont isomorphes à des sous-groupes fermés d'un produit cartésien dénombrable de groupes finis.

PROPOSITION 1.11. *Soit d un nombre premier. Alors tout pro-d-groupe à base dénombrable est isomorphe à un sous-groupe fermé de* $\mathsf{Aut}_*(\mathcal{T}_d)$.

On s'intéresse maintenant à la structure profinie qu'on peut donner à un sous-groupe G de $\mathsf{Aut}\,\mathcal{T}$.

DEFINITION 1.12. Soit G un groupe résiduellement fini. Sa *complétion profinie* $\widehat{G}$ est la limite du système projectif formé des G/N où N est un sous-groupe normal d'indice fini de G, avec une flèche naturelle $G/N \twoheadrightarrow G/M$ chaque fois que $N \subset M$. Sa *pro-p-complétion* est la limite inverse du système formé des G/N où N est un sous-groupe normal d'indice une puissance de p.

LEMMA 1.13. *Soit $G < \mathsf{Aut}\,\mathcal{T}$ doté de la propriété de congruence. Alors sa complétion profinie $\widehat{G}$ est isomorphe (en tant que groupe profini) à sa fermeture topologique $\overline{G}$ dans* $\mathsf{Aut}\,\mathcal{T}$.

Si de plus G est inclus dans un pro-p-sous-groupe de Sylow $\mathsf{Aut}_p(\mathcal{T})$ de $\mathsf{Aut}\,\mathcal{T}$, alors $\overline{G}$ est isomorphe à la pro-p-complétion $\widehat{G}_p$ de G.

DÉMONSTRATION. Par la propriété de congruence, $\{\overline{\mathsf{Stab}_G(n)}\}$ est une base de voisinages ouverts de l'identité dans $\widehat{G}$. □

DEFINITION 1.14. Soit G un sous-groupe fermé de $\mathsf{Aut}\,\mathcal{T}$. Sa *dimension de Hausdorff* $\dim(G)$ est définie dans [**BS97**] par

$$\dim(G) = \liminf_{n\to\infty} \frac{\log|G_n|}{\log|\mathsf{Aut}(\mathcal{T})_n|}.$$

Si de plus d est un nombre premier et G est contenu dans $\mathsf{Aut}_*(\mathcal{T})$, un d-Sylow de $\mathsf{Aut}\,\mathcal{T}$, sa *$*$-dimension de Hausdorff* $\dim_*(G)$ est définie par

$$\dim_*(G) = \liminf_{n\to\infty} \frac{d-1}{d^n}\log_d|G_n| = \liminf_{n\to\infty} \frac{\log_d|G_n|}{\log_d|\mathsf{Aut}_*(\mathcal{T})_n|}.$$

En particulier, la dimension de Hausdorff de $\mathsf{Aut}\,\mathcal{T}$ est 1, et elle est invariante par passage à un sous-groupe d'indice fini.

L'étude des pro-p-groupes est essentielle pour la compréhension des p-groupes, comme on le voit par exemple dans [**Sha95b**] et [**KLP97**]. Les résultats et conjectures s'expriment naturellement dans ce langage, voir par exemple la conjecture 6.10.

1.4. Groupes et automates finis

Il existe une classe d'automorphismes de $\mathcal{T}$ suffisamment générale pour faire apparaître des phénomènes intéressants : tous les p-groupes finis, des p-groupes infinis, des groupes de croissance intermédiaire, des groupes libres, etc., tout en étant assez restreinte pour autoriser leur analyse, en particulier à l'aide de démonstrations par récurrence. Il s'agit des *automorphismes automatiques*, qu'on définit plus bas.

Les automates finis trouvent de nombreuses applications en théorie des groupes, en particulier comme *accepteurs de mots*, où un automate reconnaît certains mots comme éléments du groupe, et un autre automate encode la multiplication du groupe — plus précisément, reconnaît les paires de mots (u, v) où u diffère de v par multiplication à droite par un générateur. Ce sujet est étudié exhaustivement dans [**ECH$^+$92**], qui développe la théorie des *groupes automatiques*.

On peut faire un usage complètement différent des automates comme *transducteurs* ou comme *machines séquentielles*, pour lesquels de bonnes références sont [**Eil74**, chapitre XI] ou [**GC71**]. Là, les automates sont eux-mêmes les éléments du groupe, et s'identifient à la transformation qu'ils accomplissent sur les mots. Ces groupes sont appelés *groupes à automates*, pour les distinguer des groupes définis au paragraphe précédent. Les automates à partir desquels ils sont construits sont appelés *machines de Mealy* ou *machines de Moore* (voir [**Glu61**] ou [**Bra84**, page 109]) ; ces deux notions sont équivalentes.

Nous présentons maintenant une définition légèrement limitée de *transducteur fini*. Dans la terminologie standard des références citées plus haut, ils seraient appelés des *transducteurs finis inversibles*.

Definition 1.15. Soit Σ un alphabet fini. Un *transducteur fini* sur Σ est un digraphe fini $\mathfrak{G} = (S, A)$, un étiquetage $\lambda : A \to \Sigma$ des arêtes de $\mathfrak{G}$ tel que pour tout sommet $v \in S$ la restriction de λ est une bijection entre $\{a \in A |\, \alpha(a) = v\}$ et Σ, et un étiquetage $\tau : S \to \mathfrak{S}_\Sigma$ des sommets (appelés *états*) de $\mathfrak{G}$ par $\mathfrak{S}_\Sigma$.

Un *transducteur initial* $\mathfrak{G}_q$ est un transducteur fini $\mathfrak{G}$ avec un état distingué $q \in S$.

Soit $\mathfrak{G}$ un transducteur fini, et $\{\mathfrak{G}_q\}_{q\in V}$ l'ensemble de ses transducteurs initiaux. Tout $\mathfrak{G}_q$ définit un automorphisme $\overline{\mathfrak{G}_q}$ de l'arbre $\mathcal{T}_\Sigma$ comme suit : soit $\sigma = \sigma_0 \dots \sigma_n$ un sommet de $\mathcal{T}$, et soit a l'arête de $\mathfrak{G}_q$ issue de q et étiquetée σ_0. On pose inductivement

$$\overline{\mathfrak{G}_q}(\sigma_0 \dots \sigma_n) = \tau(q)(\sigma_0)\overline{\mathfrak{G}_{\omega(a)}}(\sigma_1 \dots \sigma_n).$$

On dira que deux transducteurs initiaux $\mathfrak{G}_q$ et $\mathfrak{G}'_{q'}$ sont *équivalents* si leurs actions $\overline{\mathfrak{G}_q}$ et $\overline{\mathfrak{G}'_{q'}}$ sur $\mathcal{T}_\Sigma$ coïncident. Il existe une forme normale de tout transducteur fini, donnée par le résultat suivant :

Lemma 1.16 (théorème 4.1 du chapitre XII de [**Eil74**]). *Soit $\mathfrak{G}$ un transducteur initial fini. Alors il existe un unique transducteur initial $\mathfrak{G}'$ équivalent à $\mathfrak{G}$ et possédant un nombre minimal d'états. (On l'appelle le* transducteur minimal *associé à $\mathfrak{G}$).*

Si $\mathfrak{G}$ est un transducteur fini, on définit $G(\mathfrak{G})$ comme le groupe engendré par les $\overline{\mathfrak{G}_q}$, où q parcourt l'ensemble des états de $\mathfrak{G}$, et on appelle $G(\mathfrak{G})$ *groupe à automates*. Le résultat suivant est bien connu, et remonte au moins à Jiří Hořejš, au début des années 60 [**Hoř63**] :

Proposition 1.17. *Soient $\mathfrak{G}_q$ et $\mathfrak{G}'_{q'}$ deux transducteurs initiaux finis sur le même alphabet Σ. Alors $(\overline{\mathfrak{G}_q})^{-1}$ et $\overline{\mathfrak{G}_q} \circ \overline{\mathfrak{G}'_{q'}}$ peuvent être représentés par des transducteurs initiaux finis.*

En général, deux transducteurs différents peuvent engendrer des groupes isomorphes. Par exemple, considérons le transducteur au milieu de la figure 1.2, associé au groupe $\overline{\Gamma}$ de la section 2.1. Le groupe qu'il engendre est isomorphe (en fait, même conjugué dans $\mathsf{Aut}\,\mathcal{T}$) au groupe engendré par le transducteur de la figure 1.3, à deux états, sur le même alphabet $\Sigma = \{1, 2, 3\}$. On peut le vérifier en notant que les transformations $\tilde{t}, \tilde{a}$ satisfont les mêmes relations récursives que t, a de l'automate de la figure 1.2. Ici ε est le 3-cycle

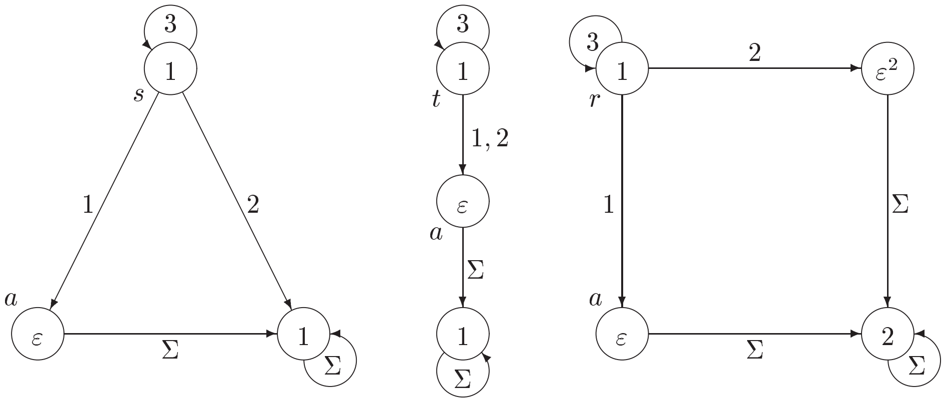


FIG. 1.2 – Les transducteurs pour les trois groupes GGS avec $A = \mathfrak{A}_3 : \Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$. Les chiffres $1, 2, 3$ sur les arêtes représentent des étiquettes dans Σ. Les symboles $1, \epsilon, \epsilon^2$ dans les cercles représentent des étiquettes dans $\mathfrak{S}_\Sigma$. Les lettres a, r, s, t représentent des choix d'état initial.

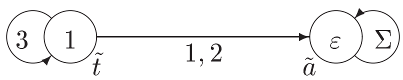


FIG. 1.3 – Le transducteur minimal associé à $\overline{\Gamma}$

$(1, 2, 3) \in \mathfrak{S}_3$. Les transducteurs $\mathfrak{G}_{\tilde{t}}$ et $\mathfrak{G}_{\tilde{a}}$ agissent ainsi :

$$\overline{\mathfrak{G}_{\tilde{t}}}(3 \dots 3\sigma_m \dots \sigma_n) = 3 \dots 3\varepsilon(\sigma_m) \dots \varepsilon(\sigma_n) \qquad \text{où } \sigma_m \neq 3,$$
$$\overline{\mathfrak{G}_{\tilde{a}}}(\sigma_0 \dots \sigma_n) = \varepsilon(\sigma_0) \dots \varepsilon(\sigma_n).$$

En revanche, pour le transducteur de la figure 1.2, on a

$$\overline{\mathfrak{G}_t}(3 \dots 3\sigma_m \dots \sigma_n) = 3 \dots 3\varepsilon(\sigma_m)\sigma_{m+1} \dots \sigma_n \qquad \text{où } \sigma_m \neq 3,$$
$$\overline{\mathfrak{G}_a}(\sigma_0 \dots \sigma_n) = \varepsilon(\sigma_0)\sigma_1 \dots \sigma_n.$$

On peut considérer les automorphismes suivants de $\mathcal{T}$:

DEFINITION 1.18. Un automorphisme g de $\mathcal{T}$ est *fini* si son portrait n'est non-trivial qu'en un nombre fini de sommets. Il est *dirigé* selon e, où e est un chemin géodésique infini, si son portrait est trivial sur e et sur les sommets à distance 2 ou plus de e. Il est *à automates* s'il existe un transducteur initial le représentant.

PROPOSITION 1.19. *Si g est un automorphisme fini, il est à automates.*

Si g est un automorphisme dirigé ; si sa direction est un mot (infini) e périodique ; et si les étiquetages à distance 1 de e forment une suite périodique, alors g est à automates.

DÉMONSTRATION. Supposons d'abord que g est fini. Soit $\mathcal{U}$ la plus petite partie connexe de $\mathcal{T}$ contenant $\emptyset$ et le support du portrait de g. Clairement $\mathcal{U}$ est un ensemble fini. On construit le transducteur initial $\mathfrak{G}$ comme suit : son ensemble de sommets est $\mathcal{U} \cap \{\infty\}$. Son ensemble d'arêtes est celui de $\mathcal{U}$, avec en plus des arêtes de tous les sommets de $\mathcal{U}$ de degré inférieur à d vers ∞, et d boucles de ∞ à lui-même. L'étiquetage des sommets est donné par le portrait de g, avec l'étiquette 1 pour le sommet ∞. Les arêtes prennent leurs étiquettes de $\mathcal{T}$, de sorte qu'en chaque sommet il y ait d arêtes montrant tous les éléments de Σ. L'état initial est $\emptyset$.

Si g est dirigé, soit e sa direction, et supposons que $e_i = e_{i+N}$ pour tout $i \in \mathbb{N}$. Supposons de plus que les étiquettes à distance 1 de e soient aussi N-périodiques. On construit le transducteur initial $\mathfrak{G}$ comme suit : il a $dN+1$ sommets, notés (x,i) pour $x \in \Sigma$ et $0 \leq i < N$ et ∞. Il y a une arête de (e_i, i) à $(x, i+1 \mod N)$ étiquetée x pour tous les $x \in \Sigma$, et d arêtes de ∞ à lui-même et de (x,i) à ∞ pour tous les $x \neq e_i$. L'étiquetage du sommet (x,i) est celui du sommet $e_0 e_1 \dots e_i x$ dans le portrait de g. L'état initial est $(e_{N-1}, 0)$. $\square$

CHAPITRE 2

Groupes à branches

Après ses premiers travaux (répondant aux questions de Burnside et Milnor), Rostislav Grigorchuk a mis en évidence une propriété fondamentale de certains groupes agissant sur des arbres : il s'agit d'une propriété d' ń auto-similarité ż, appelée *branchedness* en anglais, qu'on pourrait traduire par ń groupes à branches ż. Elle n'est apparue explicitement qu'en 1997 dans un chapitre de Grigorchuk [**Gri00**].

Definition 2.1. Soit $G < \mathsf{Aut}\,\mathcal{T}$ un groupe. G est dit *à branches faibles* si tous les stabilisateurs rigides $\mathsf{Rist}_G(\sigma)$, définis en (8), sont infinis.

G est dit *à branches* s'il est sphériquement transitif, et si $\mathsf{Rist}_G(n)$ est d'indice fini dans G pour tout n.

G est dit *à branches régulières* s'il contient un sous-groupe normal d'indice fini $K < \mathsf{Stab}_G(1)$ tel que

$$K^\Sigma < \psi(K).$$

Du fait que $\mathsf{Rist}_G(n)$ est un produit direct de d^n groupes, où $d = |\Sigma|$, on peut reformuler la définition ainsi :

Lemma 2.2. *G est à branches si et seulement s'il existe une famille $\{K_n\}$ de sous-groupes d'indice fini de G tels qu'on ait $(K_{n+1})^\Sigma < \psi(K_n)$ pour tout $n \in \mathbb{N}$.*

Lemma 2.3. *Si G est fractal et à branches régulières, alors il est à branches. Si G est à branches, alors il est à branches faibles.*

Démonstration. Supposons que G est à branches régulières sur le sous-groupe K. On peut clairement voir K^{Σ^n}, à travers ψ_n, comme un sous-groupe de $\mathsf{Rist}_G(n)$; il est d'indice dans G^{Σ^n}, donc $\mathsf{Rist}_G(n)$ est d'indice fini dans G. La deuxième affirmation suit du fait qu'un groupe à branches est sphériquement transitif, donc infini, et que ń d'indice fini dans un groupe infini ż est logiquement plus fort que ń infini ż. □

En fait, tous les exemples qu'on considérera sont à branches régulières. Cela équivaut à dire que dans le lemme 2.2 on peut supposer que tous les K_n coïncident.

On voit ainsi les diverses récursivités entrer en jeu : l'arbre Σ^* est récursif puisque ses sous-arbres $\sigma\Sigma^*$ sont isomorphes à lui-même pour tout $\sigma \in \Sigma^*$; les groupes peuvent l'être dans le sens qu'ils contiennent un sous-groupe K qui contient ń géométriquement ż un produit direct de copies de lui-même ; et le spectre de certains de ces groupes peut aussi être un ensemble fractal, voir [**BG99b**].

L'existence d'un sous-groupe K avec les propriétés ci-dessus implique ceci : soit X une transversale de K dans G et Y une transversale de K^Σ dans K. On a alors naturellement un portrait à branches de tout élément (différent du portrait défini en 1.6) : le sommet $\emptyset$ est étiqueté par un élément de $X \times Y$ et chacun des sommets suivants est étiqueté par un élément de Y. On le définit ainsi :

Definition 2.4. Soit $g \in G$ un élément d'un groupe à branches régulières. Son *portrait à branches* se définit ainsi : on écrit $g = g_\emptyset x$ avec $g_\emptyset \in K$ et $x \in X$, et on étiquette la racine par x. Ensuite on écrit inductivement $g_\sigma = (g_{\sigma 0}, g_{\sigma 1}, \dots)y$ pour des $g_{\sigma i} \in K$ (où i parcourt Σ) et $y \in Y$. On étiquette le sommet σ par y.

De nouveau, on présente des exemples pour le groupe $\mathfrak{G}$ de Grigorchuk. Ici $\mathfrak{G}/K$ est un groupe d'ordre 16 dont le centre est d'ordre 2, et $K/(K \times K)$ est un groupe cyclique d'ordre 4, engendré par $(ab)^2$.

Dans ce langage, la suite centrale descendante de $\mathfrak{G}$ peut s'exprimer ainsi : $\gamma_2(\mathfrak{G})$ contient tous les éléments dont le portrait à branches a son X-étiquette dans le centre de $\mathfrak{G}/K$ et des Y-étiquettes quelconques ; $\gamma_3(\mathfrak{G})$ contient tous les éléments dont le portrait à branches a X-étiquette triviale, Y-étiquette à la racine une puissance de $(ab)^4$, et autres Y-étiquettes quelconques ; etc. Le résultat précis pour les exemples de groupe $\mathfrak{G}$ et $\tilde{\mathfrak{G}}$ est donné dans [**BG00a**], et est reproduit dans le théorème 6.13.

Le fait d'être un groupe à branches permet d'établir des critères précis déterminant si un groupe est juste-infini, ou juste non-résoluble. On rappelle d'abord ces notions.

Definition 2.5. Un groupe G est *juste-infini* s'il est infini, mais si tous les quotients stricts sont finis ; ou, de façon équivalente, si tous les sous-groupes normaux de G sont indice fini dans G.

Un groupe G est *juste non-résoluble* s'il n'est pas résoluble, mais si tous ses quotients stricts le sont.

On note d'emblée que pour vérifier qu'un groupe G est juste-infini (respectivement juste non-résoluble), il suffit de vérifier que tous les sous-groupes normaux de la forme $\langle g\rangle^G$ sont d'indice fini (respectivement de quotient résoluble).

Proposition 2.6. *Soit G un groupe à branches régulières sur K. Alors*
- *G est juste-infini si et seulement si $|K : K'| < \infty$;*
- *G est juste non-résoluble si et seulement si $G/K_{(1)}$ est résoluble, où $K_{(1)}$ désigne $\psi(K) < G^\Sigma$.*

En particulier, si d est premier, tout sous-groupe à branches régulières du pro-d-Sylow $\mathsf{Aut}_*(\mathcal{T})$ est juste non-résoluble.

2.1. Exemples Principaux

Les premiers exemples de groupes de Burnside [**Bur02**] (c'est-à-dire de groupes infinis de type fini et de torsion) ont été trouvés par Evgeniĭ Golod en 1964 [**Gol64, Gol68**]. Depuis lors, de nombreux mathématiciens, dont Sergei Alëshin [**Ale72**], Vitaliĭ Sushchanskiĭ [**Suš79**], Alexander Ol'shanskiĭ [**Ol'82**], et Narain Gupta et Saïd Sidki [**GS83a**], ont découvert d'autres tels exemples.

On se concentre sur les groupes agissant sur un arbre régulier, correspondant à la construction d'Alëshin. Dans cette section, on décrit les exemples de groupes $\mathfrak{G}, \tilde{\mathfrak{G}}, \Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$ étudiés dans [**BG99b, BG99a**]. Seul le premier exemple, $\mathfrak{G}$, sera l'objet d'un chapitre plus approfondi.

On prend d'abord $d = 2$, et on définit les transformations suivantes : a est l'automorphisme permutant les deux branches supérieures de $\mathcal{T}_2$. On définit récursivement b comme l'automorphisme de $\mathcal{T}_2$ agissant comme a sur la branche de gauche et comme c sur la branche de droite, c comme l'automorphisme de $\mathcal{T}_2$ agissant comme a sur la branche de gauche et comme d sur la branche de droite, et d comme l'automorphisme de $\mathcal{T}_2$ agissant comme 1 sur la branche de gauche et comme b sur la branche de droite, En formules, cela revient à poser

$$\begin{aligned} b(0x\sigma) &= 0\overline{x}\sigma, & b(1\sigma) &= 1c(\sigma),\\ c(0x\sigma) &= 0\overline{x}\sigma, & c(1\sigma) &= 1d(\sigma),\\ d(0x\sigma) &= 0x\sigma, & d(2\sigma) &= 1b(\sigma), \end{aligned}$$

où on a $\overline{1} = 0$ et $\overline{0} = 1$. Notons qu'ici, pour des raisons historiques, on a gardé $\Sigma = \{0, 1\}$, et non $\{1, 2\}$ comme on le ferait selon les définitions précédentes.

$\mathfrak{G}$ est le groupe engendré par $\{a, b, c, d\}$. On vérifie instantanément que ces générateurs sont d'ordre 2 et que $\{1, b, c, d\}$ est un groupe de Klein ; on peut donc se passer d'un quelconque des générateurs parmi b, c, d.

On peut aussi définir les automorphismes $\tilde{b}, \tilde{c}, \tilde{d}$ de $\mathcal{T}_2$, par les formules

$$\begin{aligned} \tilde{b}(0x\sigma) &= 0\overline{x}\sigma, & \tilde{b}(1\sigma) &= 1\tilde{c}(\sigma), \\ \tilde{c}(0\sigma) &= 0\sigma, & \tilde{c}(1\sigma) &= 1\tilde{d}(\sigma), \\ \tilde{d}(0\sigma) &= 0\sigma, & \tilde{d}(1\sigma) &= 1\tilde{b}(\sigma). \end{aligned}$$

Le groupe $\tilde{\mathfrak{G}}$ est le groupe engendré par $\{a, \tilde{b}, \tilde{c}, \tilde{d}\}$. Clairement, ces générateurs sont encore d'ordre 2, et $\{\tilde{b}, \tilde{c}, \tilde{d}\}$ engendre un groupe abélien élémentaire d'ordre 8.

On se bornera à mentionner que $\tilde{\mathfrak{G}}$ contient $\mathfrak{G}$ comme $\langle a, \tilde{b}\tilde{c}, \tilde{c}\tilde{d}, \tilde{d}, \tilde{b}\rangle$, mais aussi des éléments d'ordre infini, comme $a\tilde{b}\tilde{c}\tilde{d}$, qui est un ń additionneur ż (voir [**BORT96**]). L'article [**BG99a**] contient bien plus d'information sur $\tilde{\mathfrak{G}}$: une présentation récursive, aussi dite *L-présentation* (voir la section 6.2), une preuve que sa croissance est sous-exponentielle, etc.

L'autre famille d'exemples qu'on considère est construite sur l'arbre $\mathcal{T}_3$, pour $d = 3$. On définit a comme l'automorphisme de $\mathcal{T}_3$ permutant les trois branches supérieures de $\mathcal{T}_3$, et les automorphismes r, s, t fixant le premier niveau Σ, et agissant sur le sous-arbre de gauche comme a, sur le sous-arbre du milieu comme $1, a, a^2$ respectivement, et sur le sous-arbre de droite comme eux-mêmes. En formules, si on adopte la convention que $\overline{1} = 2, \overline{2} = 3, \overline{3} = 1$, on a

$$\begin{aligned} r(1x\sigma) &= 1\overline{x}\sigma, & r(2x\sigma) &= 2\overline{x}\sigma, & r(3\sigma) &= 3t(\sigma), \\ s(1x\sigma) &= 1\overline{x}\sigma, & s(2x\sigma) &= 2x\sigma, & s(3\sigma) &= 3t(\sigma), \\ t(1x\sigma) &= 1\overline{x}\sigma, & t(2x\sigma) &= 2\overline{\overline{x}}\sigma, & t(3\sigma) &= 3t(\sigma). \end{aligned}$$

On définit ainsi les groupes $\Gamma = \langle a, r\rangle$, $\overline{\Gamma} = \langle a, s\rangle$ et $\overline{\overline{\Gamma}} = \langle a, t\rangle$. Le groupe Γ a été étudié par Narain Gupta et Jacek Fabrykowski [**FG85, FG91**]. Ils affirment, dans le premier article, que Γ est de croissance intermédiaire. Cette démonstration semble malheureusement incorrecte, et le deuxième article tente d'y remédier.

Le groupe $\overline{\Gamma}$ n'est pas un groupe à branches, bien qu'il soit faiblement à branches. Il est juste non-résoluble, et la structure de ses sous-groupes est relativement bien comprise — voir [**BG99a**].

Le groupe Γ a été étudié par Narain Gupta et Saïd Sidki [**GS84, GS83b, GS83a**]. Ses propriétés algébriques (en particulier, son groupe d'automorphismes) ont été bien déterminées [**Sid87a**] : c'est un 3-groupe infini, on connaît sa suite centrale descendante, etc. Saïd Sidki affirme dans [**Sid87b**] pouvoir déterminer une L-présentation (voir la section 6.2), bien qu'il n'y ait aucune formulation explicite.

On verra dans la dernière partie de ce travail que ces trois groupes sont de croissance intermédiaire.

Une généralisation de ces deux types de groupes a été introduite par Rostislav Grigorchuk dans [**Gri00**], section 8. Il s'agit des *groupes spéciaux*, dont on se borne ici à rappeler la définition (en la réduisant au cas des arbres homogènes, et non généralement sphériquement transitifs).

DEFINITION 2.7. Soit R un sous-groupe de $\mathfrak{S}_\Sigma$, soit D un ensemble fini, et soit $\{\omega^n_{x,k}\}_{x\in D, k\in\Sigma\setminus\{d\}, n\in\mathbb{N}}$ une famille d'éléments de $\mathfrak{S}_\Sigma$. Comme toujours, on fait agir $\mathfrak{S}_\Sigma$ sur Σ^* en n'agissant que sur le premier symbole des suites. Les données $\omega^n_{x,k}$ munissent D d'une action sur Σ^* comme suit : pour $x \in D$ et une suite $\sigma = d \dots d\sigma_n\tau \in \Sigma^*$, on a

$$x(d \dots d\sigma_n\tau) = d \dots d\sigma_n\omega^n_{x,\sigma_n}(\tau),$$

où σ_n est le premier symbole différent de d.

Un *groupe spécial* est un groupe de la forme $\langle R, D\rangle$, pour des ensembles R et D agissant comme spécifié ci-dessus.

CHAPITRE 3

Croissance dans les groupes

Ce chapitre sert d'introduction à la croissance des groupes et des espaces homogènes ; il sera utilisé au chapitre 4 où on étudiera la croissance des espaces homogènes associés à un sous-groupe parabolique, dans le chapitre 6 où on décrira des bornes sur la croissance du groupe de Grigorchuk $\mathfrak{G}$, et dans le chapitre 14, où on montrera des résultats de croissance pour une large famille de groupes.

Dans tout ce chapitre, on supposera implicitement que G est un groupe de type fini.

Definition 3.1. Soit G un groupe engendré par un ensemble fini S, et soit X un espace sur lequel G agit transitivement. Un *poids* est une fonction $\omega : S \to \mathbb{R}_+^*$. Le poids s'étend naturellement à une fonction $|\cdot| : G \to \mathbb{R}_+$, toujours appelée poids, par

$$|g|_\omega = \min\{\omega(s_1) + \cdots + \omega(s_n) | \, g = s_1 \dots s_n \text{ avec } s_i \in S\}.$$

La *croissance* de X en $x \in X$ est la fonction $\gamma_\omega^x : \mathbb{R}_+ \to \mathbb{N}$ définie par

$$\gamma_\omega^x(n) = |\{gx \in X | \, |g|_\omega \leq n\}|.$$

Lorsque ω n'est pas spécifié, on prend le poids standard $\omega(s) \equiv 1$ et on n'écrit que $\gamma_{G,S}^x$. La *croissance* de G est la croissance en $1 \in G$ du G-ensemble G agissant sur lui-même par multiplication à gauche.

Par exemple, la croissance de $\mathbb{Z}$ engendré par $S = \{1, -1\}$ est $\gamma_{G,S}(n) = 2n + 1$, car on peut exprimer tout entier de l'intervalle $[-n, n]$ comme un mot de longueur au plus n en S. La croissance du $\mathbb{Z}$-module $\mathbb{Z}/\ell\mathbb{Z}$, elle, est $\gamma_{G,S}^0(n) = \min\{2n + 1, \ell\}$.

On remarque que la donnée de S fini et $\omega : S \to \mathbb{R}_+^*$ équivaut à la donnée d'une fonction $\omega : G \to \mathbb{R}_+$ de support fini. C'est la raison pour laquelle on ne spécifie pas S dans la notation γ_ω^x.

3.1. Séries de croissance

La croissance d'un groupe G peut être étudiée de façon exacte. Pour ce faire, on considère généralement le poids standard $\omega(s) \equiv 1$, et on forme la série formelle

$$\Gamma_G^S(t) = \sum_{n \geq 0} \gamma_{G,S}(n) t^n \in \mathbb{Z}[[t]].$$

On peut se demander si une forme fermée existe pour les coefficients $\gamma_{G,S}(n)$ ou pour la série Γ_G^S, vue comme une fonction analytique. À titre d'exemple, on a :

Theorem 3.2 (Benson [**Ben83**]). *Soit G un groupe contenant un sous-groupe abélien d'indice fini, et engendré par un ensemble fini S. Alors la fonction $\Gamma_G^S(t)$ est rationnelle (c'est-à-dire appartient à $\mathbb{Z}(t)$).*

Ce résultat a été généralisé par Fabrice Liardet [**Lia96**].

On a aussi :

Theorem 3.3 (Gromov [**GH90**, chapitre 9]). *Soit G un groupe hyperbolique au sens de Gromov, et engendré par un ensemble fini S. Alors la fonction $\Gamma_G^S(t)$ est rationnelle.*

On ne connaît pas d'autres classes de groupes ayant une série de croissance rationnelle pour tout système de générateurs, bien qu'il soit conjecturé que ce soit le cas pour le groupe de Heisenberg $\langle x, y | \, [x, y] \text{ central}\rangle$.

Il existe aussi des exemples isolés de groupes pour lesquels la série Γ_G^S est algébrique, mais pas rationnelle [**Par92**].

3.2. Croissance asymptotique

On remarque que la fonction de croissance dépend fortement du choix de S et du poids ω ; il en va de même pour la série de croissance Γ_G^S, puisque pour certains groupes (des groupes nilpotents de rang 2) elle est rationnelle pour un certains système de générateurs, et transcendante pour un autre [**Sto96**]. Si on veut un objet intrinsèque au groupe, c'est-à-dire indépendant de S et ω, on ne peut considérer que la classe de γ_ω^x relativement à une relation d'équivalence. La notion d'équivalence la plus fine est la suivante :

DEFINITION 3.4. Soient $f, g : \mathbb{R}_+ \to \mathbb{N}$ deux fonctions. On dit que f est *dominée* par g, ce qu'on note $f \precsim g$, s'il existe une constante $C \in \mathbb{N}$ telle que $f(n) \le g(Cn)$. On dit que f et g sont *équivalentes*, noté $f \sim g$, si $f \precsim g$ et $g \precsim f$.

LEMMA 3.5. *Soit X un G-ensemble. Soient S et S' deux systèmes de générateurs finis de G, munis de poids ω et ω', et soient x, x' deux éléments de X. Alors les fonctions de croissance γ_ω^x et $\gamma_{\omega'}^{x'}$ sont équivalentes.*

DÉMONSTRATION. On pose d'abord $B = \max_{s\in S} |s|_{\omega'}/\omega(s)$. Alors on a $|g|_{\omega'} \le B|g|_\omega$, et donc $\gamma_\omega^x(n) \le \gamma_{\omega'}^x(Bn)$.

Soit maintenant $p \in G$ tel que $px' = x$, et soit $B' = |p|_{\omega'}$. Comme pour tout gx on peut écrire $gx = gpx'$, on a $\gamma_{\omega'}^x(n) \le \gamma_{\omega'}^{x'}(n + B')$.

Finalement, si on pose $M = \min\{\omega'(s) | \, s \in S'\}$, on a $\gamma_{\omega'}^x(t) = 1$ pour $t \in [0, M[$, et on peut donc écrire $\gamma_{\omega'}^x(n) \le \gamma_{\omega'}^{x'}(n(1 + B'/M))$.

En combinant les inégalités ci-dessus, on a $\gamma_\omega^x(n) \le \gamma_{\omega'}^{x'}(Cn)$, avec $C = B(1+B'/M)$, et donc $\gamma_\omega^x \precsim \gamma_{\omega'}^{x'}$. Par symétrie, on a donc $\gamma_\omega^x \sim \gamma_{\omega'}^x$. ☐

On notera désormais γ_X la classe d'équivalence des fonctions de croissance de X pour la relation $\sim$.

LEMMA 3.6. *Soit X un G-ensemble. On a alors $\gamma_X \precsim 2^n$.*

DÉMONSTRATION. Soit S un système de générateurs fini pour G, et ω le poids standard sur S. On a alors $\gamma_\omega^x(n) \le |S|^n$ pour tout $n \in \mathbb{N}$ et tout $x \in X$, et donc $\gamma_X \precsim 2^{n \log_2 |S|} \sim 2^n$. ☐

DEFINITION 3.7. Soit $f : \mathbb{R}_+ \to \mathbb{N}$ une fonction. S'il existe un entier D tel que $f \sim n^D$, on dit que f est *polynômiale* ; si $n^D \not\precsim f$ pour tout D on dit que f est *superpolynômiale*.

Si $f \sim 2^n$, on dit que f est *exponentielle*, alors que si $f \not\precsim 2^n$ on dit que f est *sous-exponentielle*.

Finalement, si f est à la fois superpolynômiale et sous-exponentielle, elle est dite *intermédiaire*.

On constate que ces notions ne dépendent que de la classe d'équivalence de f. On a les résultats suivants :

THEOREM 3.8 (Gromov [**Gro81a**]). *Si γ_G n'est pas superpolynômiale, alors γ_G est polynômiale, et G est virtuellement nilpotent.*

On a même une formule pour le degré de γ_G, due à Yves Guivarc'h et Hyman Bass [**Gui70, Bas72**] : si $\{G_n\}_{n\ge 1}$ est la suite centrale descendante de G et $d_n = \dim(G_n/G_{n+1} \otimes \mathbb{Q})$ est le rang du n-ième quotient, on a

$$\deg \gamma_G = \sum_{n\ge 1} n d_n.$$

Il est par ailleurs hautement non-trivial qu'il existe des groupes à croissance intermédiaire ; par exemple, un résultat de Jacques Tits [**Tit72**] montre qu'un groupe linéaire est soit de croissance polynômiale, soit de croissance exponentielle. Le premier groupe de croissance intermédiaire a été découvert par Rostislav Grigorchuk [**Gri83**], en réponse à une célèbre question de John Milnor [**Mil68b**].

PROPOSITION 3.9. *Soit G un groupe de type fini, H un sous-groupe de type fini de G, et G/I un G-espace homogène. Alors $\gamma_H \precsim \gamma_G$ et $\gamma_{G/I} \precsim \gamma_G$.*

Si de plus H est normal d'indice fini, alors $\gamma_G \precsim [G : H]\gamma_H$; et si I est fini, alors $\gamma_G \precsim |I|\gamma_{G/I}$.

DÉMONSTRATION. On choisit un système S de générateurs de G qui contient un système T de générateurs de H, on fixe un poids ω sur S, et on note $|\cdot|_S$ le poids induit sur G et sa projection sur G/I, et $|\cdot|_T$ le poids induit sur H. Clairement $\{g \in H| \, |g|_T \leq n\} \subset \{g \in G| \, |g|_S \leq n\}$ et $\{gI \in G/I| \, |g|_S \leq n\} \subset \{g \in G| \, |g|_S \leq n\}$, ce qui montre la première partie.

On suppose maintenant H normal d'indice fini, avec une transversale X dans G. On prend un système de générateurs T de H invariant par X, et on prend $S = T \cup X$ comme générateurs de G. Tout élément de G s'écrit alors de manière minimale sous la forme $t_1 t_2 \dots t_n x$ pour des $t_i \in T$ et $x \in X$. On a donc $\{g \in G| \, |g|_S \leq n+1\} \subset \{g \in H| \, |h|_T \leq n\} \times X$, donc $\gamma_G(n+1) \leq |X|\gamma_H(n)$.

On suppose enfin I fini. Considérons la boule $\{g \in G| \, |g|_\omega \leq n\}$. Elle se projette dans G/I avec au plus $|I|$ éléments dans chaque fibre, d'où $\gamma_G(n) \leq |I|\gamma_{G/I}(n)$. □

La proposition suivante ne sera pas utilisée, mais illustre le rapport entre les séries de croissance et les groupes à automates :

PROPOSITION 3.10. *Soit $G(\mathfrak{G})$ un groupe à automates. Pour tout $n \in \mathbb{N}$, soit $\mathfrak{G}^{(n)}$ le produit de n copies du transducteur $\mathfrak{G}$* [**GC71**]*, et soit $\delta(n)$ le nombre minimal d'états de $\mathfrak{G}^{(n)}$. Alors $\gamma_G \sim \delta$.*

DÉMONSTRATION. Les états de $\mathfrak{G}$ sont en bijection avec un système de générateurs S de $G(\mathfrak{G})$, et les états de $G(\mathfrak{G})^{(n)}$ sont décrits par S^n. Si deux mots de S^n induisent le même élément du groupe, on peut diminuer le nombre d'états de $G(\mathfrak{G})^{(n)}$ en combinant ces deux états en un seul. Ainsi, pour l'automate minimisé associé à $G(\mathfrak{G})^{(n)}$, ses états sont en bijection avec l'image de S^n dans $G(\mathfrak{G})$. □

CHAPITRE 4

Sous-groupes paraboliques

On décrit dans ce chapitre le contenu de l'article [**BG99a**], écrit avec Rostislav Grigorchuk. Il contient autant des résultats algébriques sur les exemples de groupes de la section 2.1 que des compléments à l'article [**BG99b**].

Définition 4.1. Soit G un groupe à branches. On considère son action sur l'arbre Σ^*, et on choisit un chemin géodésique (c'est-à-dire s'éloignant continuellement de la racine $\emptyset$ dans Σ^*), noté e. Le *sous-groupe parabolique* associé est le stabilisateur de e dans G, noté P_e.

La terminologie vient du fait suivant : un chemin géodésique équivaut à un point du bord de l'arbre, $\Sigma^{\mathbb{N}}$. Dans la définition habituelle, un sous-groupe P de G est parabolique s'il est le stabilisateur d'un point du bord d'un G-espace convenable, par exemple l'isotropie d'un point de $\mathbb{R} \cup \{\infty\}$ pour une action de G sur l'espace hyperbolique $\mathbb{H}$.

Clairement, par ailleurs, si on note e_n le point à distance n de $\emptyset$ sur e, on considère $P_n = \mathsf{Stab}_G(e_n)$, et on a

$$P_e = \cap_{n \in \mathbb{N}} P_n.$$

On a défini les portraits à branches en 2.4. Cette décomposition d'un élément $g \in G$ selon l'arbre correspond à une décomposition du groupe G en extensions par des groupes finis. Le résultat suivant est prouvé dans [**BG99a**] dans le cas particulier des exemples de la section 2.1.

Theorem 4.2. *Soit G un groupe fractal à branches régulières, et soit e le chemin d^∞ le plus à droite dans $\mathcal{T}$. Alors il existe une famille finie de sous-groupes $G = G_1 > \cdots > G_n = G_{n+1}$ tels qu'on ait*

$$\psi(G_i \cap \mathsf{Rist}_G(1)) = G_{i+1}^{\Sigma}$$

et des suites exactes

$$1 \longrightarrow G_{i+1}^{\Sigma} \longrightarrow G_i \longrightarrow Q_i \longrightarrow 1$$

avec quotients Q_i finis, pour tout $i \in \{1, \dots, n\}$.

Si on pose $P_i = P \cap G_i$, on a de plus

$$\psi(P_i \cap \mathsf{Rist}_G(1)) = G_{i+1}^{d-1} \times P_{i+1}$$

et des suites exactes

$$1 \longrightarrow G_{i+1}^{d-1} \times P_{i+1} \longrightarrow P_i \longrightarrow R_i \longrightarrow 1$$

avec quotients R_i finis, pour tout $i \in \{1, \dots, n\}$.

Un des intérêts de ce résultat est qu'il permet de déterminer les orbites de P sur G, et donc aussi sur G/P. On utilisera cela dans la section suivante.

Montrons toutefois d'abord la réalisation du théorème 4.2 pour le groupe $\mathfrak{G}$. On a

$$\begin{aligned}
&1 \longrightarrow B \times B \longrightarrow G \longrightarrow \langle a, c \rangle \longrightarrow 1,\\
&1 \longrightarrow K \times K \longrightarrow B \longrightarrow \langle b, b^a \,|\, (ab)^8 \rangle \longrightarrow 1,\\
&1 \longrightarrow K \times K \longrightarrow K \longrightarrow \langle (ab)^2 \,|\, (ab)^8 \rangle \longrightarrow 1.
\end{aligned}$$

La décomposition de P est similaire : si on pose $P = \mathsf{Stab}_G(1^\infty)$, et $Q = P \cap B$, $R = P \cap K$, on a

$$1 \longrightarrow B \times Q \longrightarrow P \longrightarrow \langle c, (ac)^4 \rangle \longrightarrow 1,$$
$$1 \longrightarrow K \times R \longrightarrow Q \longrightarrow \langle b, (ac)^4 \rangle \longrightarrow 1,$$
$$1 \longrightarrow K \times R \longrightarrow R \longrightarrow \langle (ac)^4 \rangle \longrightarrow 1.$$

Definition 4.3. Soit G un groupe et H un sous-groupe. H est *faiblement maximal* si H est d'indice infini dans G, mais s'il n'existe pas de sous-groupe d'indice infini dans G contenant strictement H.

On montre alors :

Proposition 4.4. *Soit G un groupe à branches et soit P un sous-groupe parabolique. Alors P est faiblement maximal.*

La croissance du G-espace P est aussi déterminée comme suit :

Definition 4.5. Deux suites infinies $\sigma, \tau \in \Sigma^{\mathbb{N}}$ sont *confinales* s'il existe un $N \in \mathbb{N}$ tel que $\sigma_n = \tau_n$ pour tout $n \geq N$.

La confinalité est une relation d'équivalence, et les classes d'équivalence sont appelées *classes de confinalité*.

Le résultat suivant est dû à Volodymyr Nekrashevych et Vitaliĭ Sushchanskiĭ :

Proposition 4.6 ([**NS99**]). *Soit G un groupe agissant sur l'arbre régulier $\mathcal{T}_\Sigma$, et supposons que pour tout élément g de G (ou, de façon équivalente, pour tout générateur) et toute suite infinie $\sigma \in \mathcal{T}_\Sigma$, les suites σ et $g\sigma$ diffèrent seulement en un nombre fini d'endroits. Alors les classes de confinalité de l'action de G sur $\partial\mathcal{T}_\Sigma$ sont des unions d'orbites. Si de plus $\mathsf{Stab}_G(\sigma)$ contient l'automorphisme enraciné a permutant les d sous-arbres sous σ pour tout $\sigma \in \mathcal{T}$, alors les orbites de l'action sont les classes de confinalité.*

La preuve du résultat suivant apparaît dans [**BG99b**] :

Proposition 4.7. *Soit $G < \mathsf{Aut}\,\mathcal{T}$ un groupe satisfaisant les conditions de la proposition 4.6, et supposons que pour l'application $\psi : g \mapsto (g_1, \dots, g_d)$ définie en (6) il y a des constantes λ, μ avec $\mu + 2\lambda < 2$ telles que $|g_i| \leq \lambda|g| + \mu$ pour tout $i \in \{1, \dots, d\}$. Soit P un sous-groupe parabolique. Alors G/P, en tant que G-ensemble, est de croissance polynômiale de degré au plus $\log_{1/\lambda}(d)$. Si de plus G est sphériquement transitif et λ est minimal pour la propriété donnée, alors la fonction de croissance $\gamma^P_{G/P}(n)$ est équivalente à $n^{\log_{1/\lambda}(d)}$.*

Ces notions sont aussi valables pour les graphes de Schreier associés aux espaces homogènes :

Definition 4.8. Soit G un groupe engendré par un système S et P un sous-groupe. Le *graphe de Schreier* de (G, P) relativement à S est le graphe $\mathcal{S}(G, P, S) = (E, V)$, où

$$V = G/P$$

et

$$E = V \times S, \quad \alpha(v, s) = v, \quad \omega(v, s) = sv.$$

Au cas où P est normal dans G, il s'agit du graphe de Cayley de G/P. En règle générale, $\mathcal{S}(G, P, S)$ est un graphe $|S|$-régulier, mais son groupe d'automorphismes n'est pas transitif sur ses sommets.

On sait, en revanche, que ń presque tous ż les graphes transitifs sont des graphes de Schreier. Plus précisément :

Theorem 4.9 ([**Lub95a**]). *Soit $\mathfrak{G}$ un graphe régulier. Alors il existe un groupe G engendré par un ensemble S et un sous-groupe P tel que $\mathfrak{G} \cong \mathcal{S}(G, P, S)$ si et seulement si $\mathfrak{G}$ contient un 1-facteur.*

Cette dernière condition est vérifiée dès que $\mathfrak{G}$ est de degré pair ou ne contient pas d'isthme.

4.1. Algèbres de Hecke

On décrira au chapitre suivant le spectre de l'opérateur de type Hecke de la représentation quasi-régulière de G sur l'espace $\ell^2(G/P)$. Plus généralement, on note $\rho_{G/H}$ la représentation de G sur l'espace $\ell^2(G/H)$.

DEFINITION 4.10. Le *commensurateur* d'un sous-groupe H de G est

$$\mathrm{comm}_G(H) = \{g \in G \mid H \cap H^g \text{ est d'indice fini dans } H \text{ et } H^g\}.$$

En d'autres termes, si on fait agir H à gauche sur les classes à droite $\{gH\}$, on a

$$\mathrm{comm}_G(H) = \{g \in G \mid H \cdot (gH) \text{ et } H \cdot (g^{-1}H) \text{ sont des orbites finies}\}.$$

PROPOSITION 4.11. *Si G est fractal et sphériquement transitif, alors $\rho_{G/P}$ est irréductible.*

On montre ce résultat en appliquant le critère suivant dû à George Mackey, c'est-à-dire en montrant que $\mathrm{comm}_G(P) = P$.

THEOREM 4.12 (Mackey [**Mac76, BH97**]). *Soit G un groupe infini et soit H un sous-groupe de G. Alors $\rho_{G/H}$ est irréductible si et seulement si $\mathrm{comm}_G(H) = H$.*

On remarque maintenant que la représentation $\rho_{G/P}$ est approximée par les représentations ρ_{G/P_n}, dans le sens que pour tout $g \neq 1$ dans G il existe un $n \in \mathbb{N}$ avec $\rho_{G/P_n}(g) \neq 1$. (Cette notion de convergence de représentations est celle utilisée dans [**GŻ97**]).

Ces représentations ρ_{G/P_n} sont de dimension finie : on peut identifier l'espace homogène G/P_n à Σ^n, l'orbite de G au niveau n, donc ρ_{G/P_n} est de dimension d^n. On va étudier sa décomposition en irréductibles.

DEFINITION 4.13. Soit G un groupe et H un sous-groupe. L'*algèbre de Hecke* (aussi appelée l'*algèbre d'entrelacement*) $\mathcal{H}(G,P)$ est l'algèbre $\mathrm{End}_{\ell^2(G)}(\ell^2(G/H)$. Elle peut être vue comme l'algèbre des fonctions (H,H)-biinvariantes sur G, avec le produit de convolution

$$(f \cdot g)(x) = \sum_{y \in G} f(xy)g(y^{-1}).$$

On peut donc penser à l'algèbre de Hecke comme à l'algèbre engendrée par les doubles classes dans $\ell^2(G)$. Le résultat suivant montre l'importance de l'algèbre de Hecke dans l'étude de la décomposition des représentations :

THEOREM 4.14 ([**CR90**], section 11D). *Soit G un groupe et H un sous-groupe. Alors $\mathcal{H}(G,H)$ est une algèbre semi-simple. Il y a une bijection canonique entre les composantes irréductibles de $\rho_{G/H}$ et les facteurs simples de $\mathcal{H}(G,H)$, qui ont les mêmes multiplicités.*

DEFINITION 4.15. Soit G un groupe et H un sous-groupe. La paire (G,H) est une paire de Gel'fand si toutes les sous-représentations irréductibles de $\rho_{G/H}$ ont multiplicité 1.

Il est clair que (G,H) est une paire de Gel'fand si et seulement si $\mathcal{H}(G,H)$ est abélienne (voir par exemple [**CR90**, exercice 18, page 306]). De plus, si c'est le cas, la décomposition de $\mathcal{H}(G,H)$ en facteurs simples est en bijection avec les doubles classes HgH de H dans G.

PROPOSITION 4.16. *Pour les cinq groupes de la section 2.1, le sous-groupe P_n a $(d-1)n+1$ orbites dans Σ^n : toutes celles de la forme $d^i x\Sigma^{n-1-i}$ pour un $x \in \Sigma \setminus \{d\}$ et un $i \in \{0,\dots,n-1\}$, et le point fixe d^n.*

On a ainsi $\dim \mathcal{H}(G,P_n) = (d-1)n+1$. De plus, $\mathcal{H}(G,P_{n-1})$ est un sommant direct de $\mathcal{H}(G,P_n)$, donc son orthogonal est un sommant de dimension $d-1$, qui est abélien car de dimension 1 ou 2 (on rappelle que $\mathcal{H}(G,P_n)$ est semi-simple). On a ainsi montré :

THEOREM 4.17. *Pour les cinq groupes de la section 2.1, les paires (G,P_n) sont de Gel'fand. La décomposition de ρ_{G/P_n} en irréductibles contient $(d-1)n+1$ composantes, dont d de dimension 1 et $d-1$ de dimension d^i, pour tous les $i \in \{1,\dots,n-1\}$.*

4.2. Congruence Quantitative

Une autre nouveauté de l'article [**BG99a**] est l'usage systématique d'une propriété pour déterminer la structure de sous-groupes d'indice fini d'un groupe branché G en se ramenant à l'étude d'un groupe fini.

Si un groupe G a la propriété de congruence, il peut être ń bien approximé ż par ses quotients $G/\operatorname{\mathsf{Stab}}_G(n)$, dans le sens que pour tout voisinage fini de l'origine dans G on peut trouver un n suffisamment grand pour que ce voisinage apparaisse isomorphiquement dans $G/\operatorname{\mathsf{Stab}}_G(n)$.

On peut donner un sens précis au ń n suffisamment grand ż de la remarque ci-dessus :

Proposition 4.18 (ń congruence quantitative ż). *Soit G un groupe à branches régulières sur son sous-groupe K, engendré par le système fini S et ayant la propriété de congruence. Soit $\partial(g)$ la profondeur du S-portrait de $g \in G$ (voir la définition 1.6).*

Soit n le plus petit entier tel qu'on ait $\langle s\rangle^G \geq \operatorname{\mathsf{Stab}}_G(n)$ pour tous les générateurs $s \in S$. Soit m minimal tel que K' contienne $\operatorname{\mathsf{Stab}}_G(m)$.

Alors pour tout sous-groupe normal non-trivial N de G, et pour n'importe quel $g \in N$ non trivial, N contient $\operatorname{\mathsf{Stab}}_G(\partial(g) + m + n)$.

L'utilisation principale de ce résultat est le suivant : on veut connaître l'indice d'un certain sous-groupe $N = \langle g\rangle$. Une fois qu'on a déterminé les constantes m et n pour le groupe G, on n'a qu'à calculer $\partial(g)$ pour trouver un $\operatorname{\mathsf{Stab}}_G(p)$ contenu dans N. On sait alors que l'indice de N dans G est le même que celui de N dans $G/\operatorname{\mathsf{Stab}}_G(p)$, qui est un groupe fini ; encore mieux, les treillis de sous-groupes entre G et N et entre $G/\operatorname{\mathsf{Stab}}_G(p)$ et N sont isomorphes.

On peut donc se servir d'un programme de théorie des groupes, comme par exemple Gap [**S$^+$93**], pour obtenir explicitement ce treillis. On donnera au chapitre 6 (dans la table 6.1) une partie du treillis des sous-groupes normaux d'indice fini dans $\mathfrak{G}$. Remarquons que l'application du théorème 4.2 à $\mathfrak{G}$ a été réalisée grâce à cette méthode.

CHAPITRE 5

Spectres de représentations

Les propriétés d'autosimilitude ne se limitent pas au groupe G et à l'arbre : on en voit aussi dans les « spectres de représentations », comme suit :

DÉFINITION 5.1. Soit $\pi : G \to U(\mathcal{H})$ une représentation unitaire du groupe G dans l'espace de Hilbert $\mathcal{H}$, et soit S un système de générateurs symétrique et fini de G. Le *spectre* de la représentation π (pour le système de générateurs S) est le spectre de l'opérateur $\sum_{s \in S} \pi(s)$ sur $\mathcal{H}$.

On peut même pousser plus loin l'analyse du spectre en considérant la *mesure spectrale* associée à l'opérateur $M = \sum_{s \in S} \pi(s)$ sur $\mathcal{H}$. Rappelons que puisque M est un opérateur symétrique (et donc normal), il admet une décomposition spectrale, c'est-à-dire que pour tous les $x, y \in \mathcal{H}$ il existe une fonction $\sigma_{x,y} : \mathbb{R} \to \mathbb{R}$ telle que

$$\langle M^n x, y \rangle = \int_{\mathbb{R}} t^n d\sigma_{x,y}(t)$$

pour tout $n \in \mathbb{N}$. Par exemple, si $\mathcal{H}$ est de dimension finie N, M est diagonalisable avec valeurs propres $\lambda_1, \dots, \lambda_N$ et vecteurs propres $v_1, \dots, v_N$, et la fonction $\sigma_{x,y}$ est constante partout, sauf en les λ_i où elle augmente de $\langle x, v_i \rangle \langle y, v_i \rangle$. Plus généralement, les discontinuités de $\sigma_{x,y}$ ne peuvent se produire qu'aux valeurs propres de M.

Dans le cadre d'une représentation quasi-régulière de G sur $\ell^2(G/H)$, la mesure spectrale que l'on calcule est $\sigma_{\delta,\delta}$, où δ est la masse de Dirac en H, c'est-à-dire la fonction valant 1 sur la classe de H et 0 ailleurs.

Rostislav Grigorchuk et l'auteur calculent le spectre de la représentation quasi-régulière $\rho_{G/P}$ de plusieurs groupes à branches dans [**BG99b**], où G est un groupe à branches et P un sous-groupe parabolique (voir le chapitre précédent). Ils déterminent aussi la mesure spectrale dans ces cas-là.

Le calcul du spectre se fait ainsi : considérons le groupe Γ de Narain Gupta et Saïd Sidki (voir la section 2.1). Il est engendré par deux transformations a et t, où a est la permutation cyclique des trois branches supérieures de $\mathcal{T}_3$, et t est défini inductivement comme fixant les trois sommets du premier niveau, et agissant comme a, a^{-1} et t sur les sous-arbres correspondants.

Si on note $L^2(\partial\mathcal{T})$ l'espace de fonctions sur le bord de l'arbre de carré intégrable pour la mesure de Bernouilli, on a $L^2(\partial\mathcal{T}) \cong L^2(\partial\mathcal{T})^3$ par l'isomorphisme ψ, et on peut écrire tout opérateur sur $L^2(\partial\mathcal{T})$ comme une matrice 3×3. Cela donne

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & r \end{pmatrix}.$$

On veut à présent calculer le spectre de $a + a^{-1} + t + t^{-1}$. Plutôt que d'attaquer directement ce problème, on considère les quotients finis Γ_n de Γ, agissant sur Σ^n, et on pose

$$Q_n(\lambda, \mu) = \det\left(t + t^{-1} + \lambda(a + a^{-1}) - \mu\right),$$

où a, t sont des matrices $3^n \times 3^n$.

Posons

$$\alpha = 2 - \mu + \lambda, \qquad \beta = 2 - \mu - \lambda,$$
$$\gamma = \mu^2 - \lambda^2 - \mu - 2, \qquad \delta = \mu^2 - \lambda^2 - 2\mu - \lambda.$$

On montre par récurrence que

$$Q_0(\lambda, \mu) = 2 + 2\lambda - \mu = \alpha + \lambda,$$
$$Q_1(\lambda, \mu) = (2 + 2\lambda - \mu)(2 - \lambda - \mu)^2 = (\alpha + \lambda)\beta^2,$$
$$Q_n(\lambda, \mu) = (\alpha\beta\gamma^2)^{3^{n-2}} Q_{n-1}\left(\frac{\lambda^2\beta}{\alpha\gamma}, \mu + \frac{2\lambda^2\delta}{\alpha\gamma}\right) \quad (n \geq 2).$$

Considérons maintenant les formes quadratiques

$$H_\theta = \mu^2 - \lambda\mu - 2\lambda^2 - 2 - \mu + \theta\lambda,$$

et la fonction $F : [-4, 5] \to [-4, 5]$ donnée par

$$F(\theta) = 4 - 2\theta - \theta^2.$$

Soit $X_2 = \{-1\}$, et itérativement $X_n = F^{-1}(X_{n-1})$ pour tous les $n \geq 3$. Remarquons que $|X_n| = 2^{n-2}$. Pour tout $n \geq 2$ on a la factorisation

$$|Q_n(\lambda, \mu)| = (2 + 2\lambda - \mu)(2 - \lambda - \mu)^{3^{n-1}+1} \prod_{\substack{2 \leq m \leq n \\ \theta \in X_m}} H_\theta^{3^{n-m}+1}. \tag{9}$$

Ainsi, d'après le résultat précédent, le spectre de G_n est formé de deux droites et d'une famille de $2^{n-1} - 1$ hyperboles H_θ avec $\theta \in X_2 \sqcup X_3 \sqcup \cdots \sqcup X_n$. Le spectre de Γ est obtenu en résolvant $Q_n(1, \mu) = 0$, ce qu'on fait maintenant.

Theorem 5.2. *Le spectre de π pour le groupe Γ est la clôture de l'ensemble*

$$\left\{ \begin{array}{rl} 4 & (\mu = 0) \\ 1 & (\mu = \frac{1}{3}) \\ 1 \pm \sqrt{6} & (\mu = \frac{1}{9}) \\ 1 \pm \sqrt{6 \pm \sqrt{6}} & (\mu = \frac{1}{3^3}) \\ 1 \pm \sqrt{6 \pm \sqrt{6 \pm \sqrt{6}}} & (\mu = \frac{1}{3^4}) \\ \dots & \end{array} \right\}.$$

Il est l'union d'un ensemble de Cantor de mesure de Lebesgue nulle symétrique autour de 1 et d'une famille dénombrable de points d'accumulation supportant toute la mesure spectrale empirique, qui a les valeurs indiquées en μ.

Corollary 5.3. *Il existe un graphe 4-régulier dont le spectre est l'ensemble décrit ci-dessus.*

Ce graphe n'est rien d'autre que le graphe de Schreier de Γ/P, pour un sous-groupe parabolique P. Il peut être décrit comme un ń graphe de substitutions ż. Une approximation en est représentée à la figure 5.3. C'est un objet fractal de dimension $\log_2(3)$, comme le montre la proposition suivante :

Proposition 5.4. *La série de croissance du graphe de Schreier $\mathcal{S}(\Gamma, P, S)$ est*

$$F(t) = \sum_{n \in \mathbb{N}} \gamma(n) t^n = \prod_{i \in \mathbb{N}} (1 + 2t^{2^i}).$$

C'est une fonction transcendante, dont le terme général $\gamma(n)$ a croissance asymptotique $n^{\log_2(3)}$.

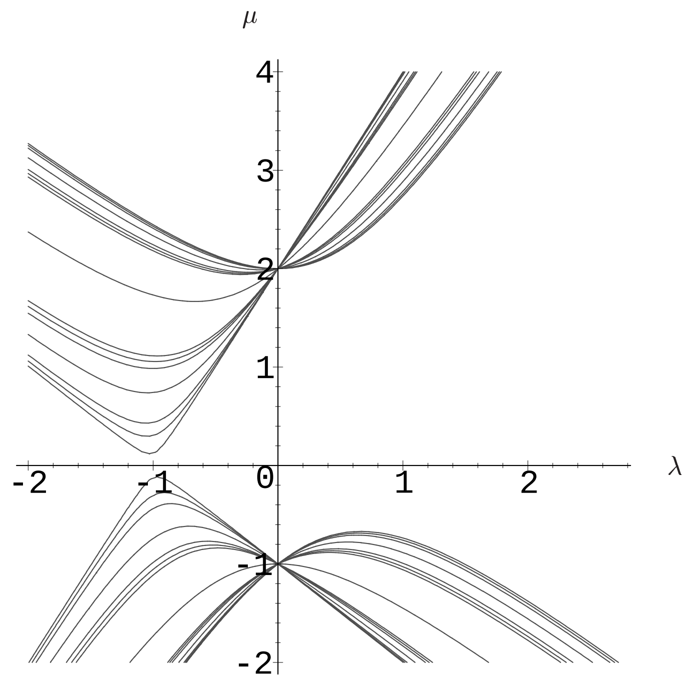


Fig. 5.1 – Les zéros de $Q_n(\lambda, \mu)$ pour Γ

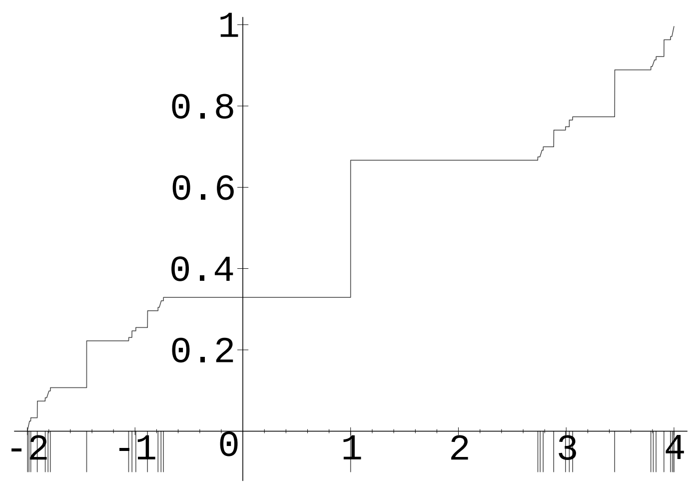


Fig. 5.2 – La mesure spectrale associée à Γ

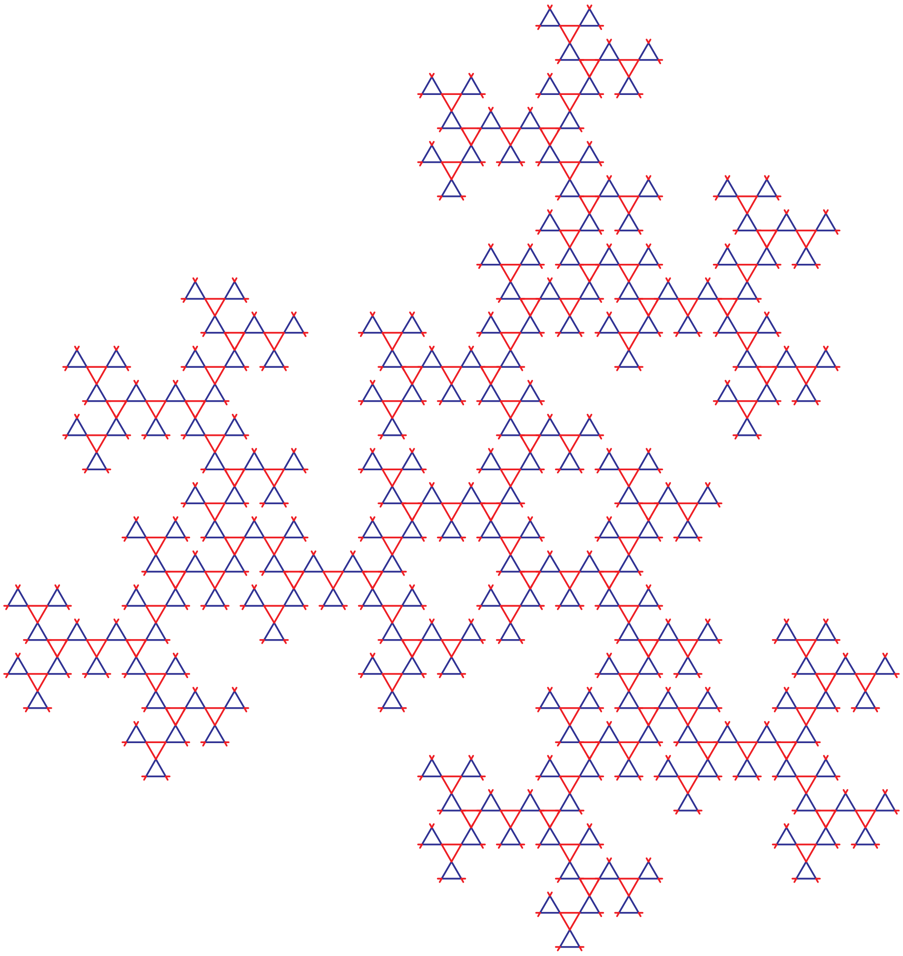


FIG. 5.3 – Le graphe de Schreier de Γ_6. Les arêtes rouges et bleues représentent les générateurs s et a.

CHAPITRE 6

Le premier groupe de Grigorchuk, $\mathfrak{G}$

On décrit ici le premier exemple de groupe à branches, conçu par Grigorchuk en 1980 comme réponse au problème de Burnside généralisé (ń existe-t-il des groupes infinis de type fini dont tous les éléments sont d'ordre fini ? ż). Il a servi ensuite à bien d'autres choses, comme réponse à des questions de croissance [**Mil68b**], de classification de groupes juste-infinis [**BG00a**], etc. Une référence précieuse pour ce groupe $\mathfrak{G}$ est le chapitre 8 de [**Har00**] ; voir aussi [**CMS98**].

Parmi les résultats importants concernant $\mathfrak{G}$, dont certains seront élaborés par la suite :

- $\mathfrak{G}$ est un groupe de type fini, infini et de 2-torsion ;
- $\mathfrak{G}$ est fractal, et est à branches régulières sur son sous-groupe K défini ci-dessous ;
- $\mathfrak{G}$ est de croissance intermédiaire, et est donc moyennable ;
- $\mathfrak{G}$ est résiduellement fini, et plus précisément admet une suite naturelle de quotients $\mathfrak{G}_n = \mathfrak{G}/\operatorname{\mathsf{Stab}}_{\mathfrak{G}}(n)$, d'ordre $2^{5\cdot 2^{n-3}+2}$ pour $n \geq 3$ (et d'ordre 2^{2^n-1} pour $n \leq 3$) l'approximant ;
- $\mathfrak{G}$ agit fidèlement sur $\mathfrak{G}/P = \mathfrak{G}/\operatorname{\mathsf{Stab}}_{\mathfrak{G}}(1^\infty)$, qui est un espace homogène de croissance linéaire (voir la section 6.3) ;
- $\mathfrak{G}$ agit fidèlement sur le bord $\partial\mathcal{T}$ de l'arbre $\mathcal{T}_2$, et ses orbites sont les classes de confinalité (voir la proposition 4.6) ;
- $\mathfrak{G}$ admet tout 2-groupe fini comme quotient ;
- $\mathfrak{G}$ est de largeur finie (voir le théorème 6.13) ;
- $\mathfrak{G}$ est juste-infini ;
- $\mathfrak{G}$ n'a pas de présentation finie, mais a une L-présentation finie (voir le théorème 6.3).

6.1. Quelques sous-groupes de $\mathfrak{G}$

On introduit les sous-groupes suivants de $\mathfrak{G}$:

$$K = \langle (ab)^2 \rangle^{\mathfrak{G}}, \qquad L = \langle (ac)^2 \rangle^{\mathfrak{G}}, \qquad M = \langle (ad)^2 \rangle^{\mathfrak{G}},$$
$$\overline{B} = \langle B, L \rangle, \qquad \overline{C} = \langle C, K \rangle, \qquad \overline{D} = \langle D, K \rangle,$$
$$T = K^2 = \langle (ab)^4 \rangle^{\mathfrak{G}},$$
$$T_{(m)} = \underbrace{T \times \cdots \times T}_{2^m}, \qquad K_{(m)} = \underbrace{K \times \cdots \times K}_{2^m}, \qquad N_{(m)} = T_{(m-1)} K_{(m)}.$$

Theorem 6.1. *On a :*

– *Les stabilisateurs restreints satisfont*

$$\operatorname{\mathsf{Rist}}_{\mathfrak{G}}(n) = \begin{cases} D & si\ n = 1, \\ K_{(n)} & si\ n \geq 2. \end{cases}$$

– *Les stabilisateurs de niveau satisfont*

$$\mathsf{Stab}_{\mathfrak{G}}(n) = \begin{cases} H & \textit{si } n = 1, \\ \langle D, T \rangle & \textit{si } n = 2, \\ \langle N_{(2)}, (ab)^4(adabac)^2 \rangle & \textit{si } n = 3, \\ \underbrace{\mathsf{Stab}_{\mathfrak{G}}(3) \times \cdots \times \mathsf{Stab}_{\mathfrak{G}}(3)}_{2^{n-3}} & \textit{si } n \geq 4. \end{cases}$$

– *Les suites dérivées de* $\mathfrak{G}$ *et* K *satisfont* $K^{(n)} = \mathsf{Rist}_{\mathfrak{G}}(2n)$ *pour tout* $n \geq 2$*, et* $\mathfrak{G}^{(n)} = \mathsf{Rist}_{\mathfrak{G}}(2n-3)$ *pour tout* $n \geq 3$*.*
– *Il y a pour tout* $\sigma \in \Sigma^n$ *une surjection* $\cdot_{|\sigma} : \mathsf{Stab}_{\mathfrak{G}}(n) \twoheadrightarrow \mathfrak{G}$ *donnée par la projection sur le facteur indicé par* σ*.*

Les premiers trois points du théorème ont été prouvés par Alexander Rozhkov dans [**Roz96b**]. Les autres ont été obtenus par la méthode de la ń congruence quantitative ż décrite à la section 4.2.

6.2. L-présentations

Igor Lysënok a découvert une présentation remarquable de $\mathfrak{G}$ par générateurs et relations [**Lys85**]. Avant de la donner, on définit les présentations de cette sorte :

Definition 6.2. Une *L-présentation* d'un groupe G est la donnée d'ensembles S et R, et d'une famille Φ d'applications $\phi : S \to S^*$. On la note

$$G = \langle S | \, R | \, \Phi \rangle.$$

Elle est *finie* si S, R et Φ sont finis.

Chaque $\phi \in \Phi$ induit un homomorphisme de monoïdes $S^* \to S^*$ par concaténation. Le groupe associé à une L-présentation est

$$G = \langle S | \, \phi_1 \dots \phi_n(r) \text{ pour tous les } \phi_i \in \Phi \text{ et } r \in R \rangle.$$

Clairement, les groupes de L-présentation finie contiennent les groupes de présentation finie. De plus, tous les $\phi \in \Phi$ induisent des endomorphismes de G. Le vocabulaire est autant un hommage à Igor Lysënok qu'une référence aux *systèmes L* (voir [**RS80**]) utilisés pour modéliser des phénomènes de croissance biologique.

Soit $S = \{a, b, c, d\}$ engendrant $\mathfrak{G}$, et soit σ la substitution sur S^* définie par

$$\sigma(a) = aca, \quad \sigma(b) = d, \sigma(c) = b, \sigma(d) = c.$$

Il est remarquable que cette substitution apparaisse dans des situations fort diverses : le résultat qui suit, la structure des espaces homogènes $\mathfrak{G}/P$ (voir la section 6.3), des algèbres de Lie associées à $\mathfrak{G}$ (voir la section 6.4), etc. On peut déjà dire que si σ induit un endomorphisme de $\mathfrak{G}$ (ce qui est montré par le théorème ci-dessous), alors on a

$$\psi(\sigma(g)) = (g', g)$$

pour un $g' \in \langle a, d \rangle \cong D_8$.

Theorem 6.3 ([**Lys85**]). *$\mathfrak{G}$ admet la L-présentation finie suivante :*

$$\mathfrak{G} = \langle a, c, d | \, a^2, (ad)^4, (adacac)^4 | \, \sigma \rangle.$$

Rostislav Grigorchuk a par ailleurs montré dans [**Gri99**] que l'ensemble des relateurs obtenus à partir de a^2, $(ad)^4$ et $(adacac)^4$ est minimal, c'est-à-dire que si l'on supprime l'un seul de ces relateurs on obtient un groupe différent. Il obtient ainsi la 2-cohomologie de $\mathfrak{G}$.

Rostislav Grigorchuk et l'auteur déterminent une L-présentation pour le groupe $\tilde{\mathfrak{G}}$ dans [**BG99a**] :

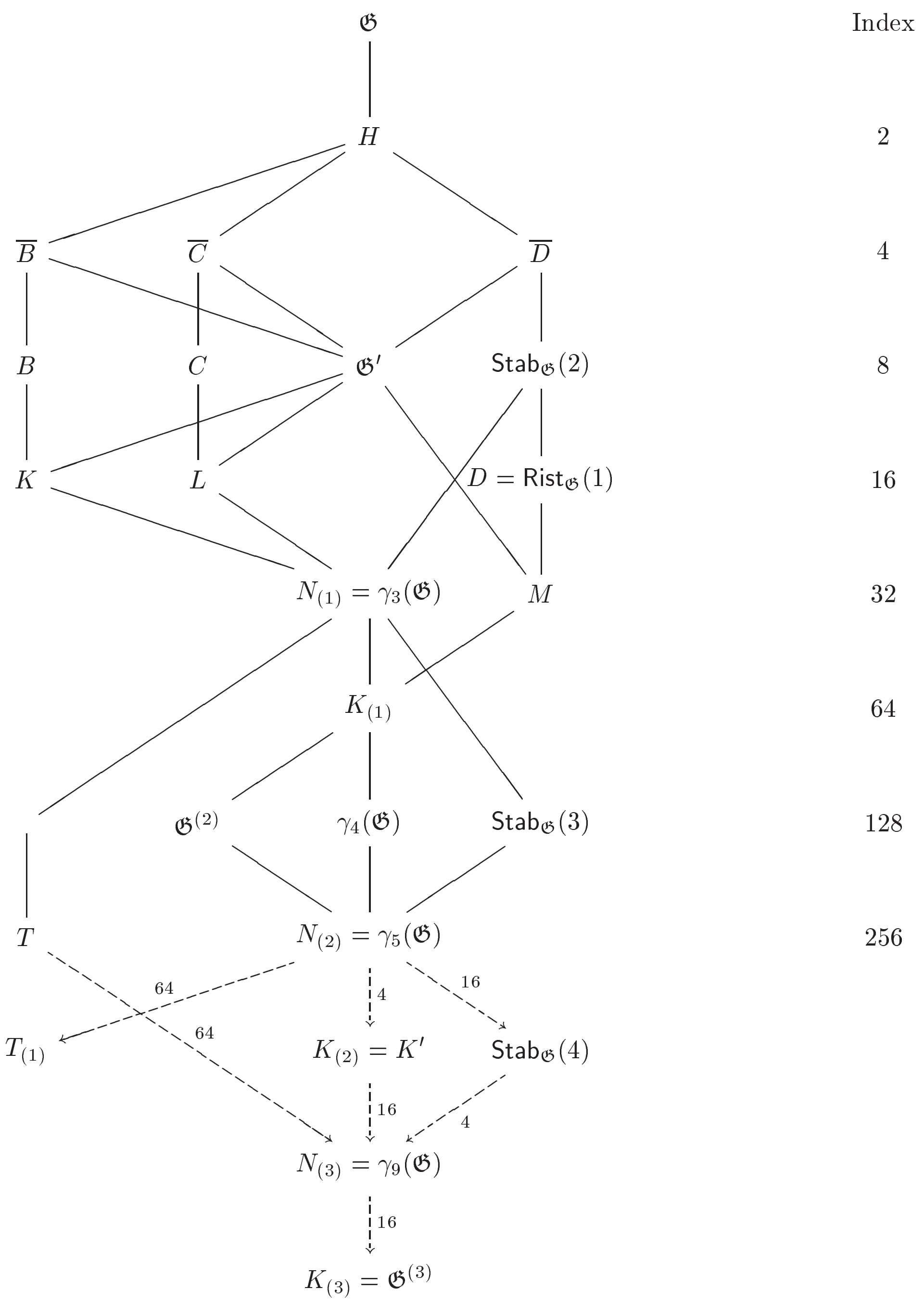


FIG. 6.1 – Le sommet du treillis des sous-groupes normaux de $\mathfrak{G}$ au-dessous de H. Les nombres à côté des arêtes en trait tillé indiquent l'indice du sous-groupe.

THEOREM 6.4. *$\tilde{\mathfrak{G}}$ admet la L-présentation finie suivante :*

$$\tilde{\mathfrak{G}} = \Big\langle a, \tilde{b}, \tilde{c}, \tilde{d} \Big| \, a^2, [\tilde{b}, \tilde{c}], (a\tilde{c})^4, (a\tilde{d})^4,$$
$$(a\tilde{c}a\tilde{d})^2, (a\tilde{b}a\tilde{b}a\tilde{c})^4, (a\tilde{b}a\tilde{b}a\tilde{d})^4, (a\tilde{b}a\tilde{b}a\tilde{c}a\tilde{b}a\tilde{b}a\tilde{d})^2 \Big| \, \tilde{\sigma} \Big\rangle,$$

où $\tilde\sigma$ est la substitution sur $\{a,\tilde b,\tilde c,\tilde d\}^*$ définie par

$$a \mapsto a\tilde b a, \quad \tilde b \mapsto \tilde d, \tilde c \mapsto \tilde b, \tilde d \mapsto \tilde c.$$

On ne sait pas si les groupes $\Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$ possèdent une L-présentation finie ; Saïd Sidki l'affirme pour $\overline{\overline{\Gamma}}$ dans [**Sid87b**], sans pour autant donner de forme explicite pour R et Φ.

Un des intérêts des L-présentations est la version suivante, explicite, du théorème de Higman :

THEOREM 6.5. *Soit G un groupe de L-présentation finie. Alors il existe un groupe $\widehat{G}$ de présentation finie contenant G. De plus, $\widehat{G}$ est obtenu de G par un nombre fini d'extensions HNN.*

DÉMONSTRATION. Soit $\langle S|R|\Phi\rangle$ une L-présentation finie de G. On prend un alphabet X_Φ en bijection avec Φ, et on pose

$$\widehat{G} = \langle S \cup X_\Phi \,|\, R \cup \{x_\phi s = \phi(s) x_\phi\}_{s\in S, x_\phi \in X_\phi}\rangle.$$

□

Claas Röver a inclu le groupe de Grigorchuk $\mathfrak{G}$ dans des groupes de présentation finie par à l'aide de techniques radicalement différentes [**Röv99a**].

6.3. L'espace homogène $\mathfrak{G}/P$

On a défini au chapitre 4 les sous-groupes paraboliques. On peut donner une description explicite de l'espace homogène $\mathfrak{G}/P$ comme suit, en remarquant qu'il s'agit d'un graphe de Schreier.

Soit $S = \{a,b,c,d\}$ engendrant $\mathfrak{G}$, soit $P = \mathsf{Stab}_{\mathfrak{G}}(1^\infty)$ le sous-groupe parabolique de $\mathfrak{G}$ associé au rayon le plus à droite de $\mathcal{T}$, et soit $P_n = \mathsf{Stab}_{\mathfrak{G}}(1^n)$. Clairement, P_n est d'indice 2^n dans $\mathfrak{G}$ et donc $\mathcal{S}(\mathfrak{G}, P_n, S)$ a 2^n sommets. De plus, du fait que $\mathfrak{G}$ a la propriété de congruence, $\mathcal{S}(\mathfrak{G}, P, S)$ est approximé par les $\mathcal{S}(\mathfrak{G}, P_n, S)$ dans le sens que toute boule dans $\mathcal{S}(\mathfrak{G}, P, S)$ se trouve isomorphiquement dans $\mathcal{S}(\mathfrak{G}, P_n, S)$ pour n assez grand.

Pour décrire les arêtes de $\mathcal{S}(\mathfrak{G}, P_n, S)$, on étend l'alphabet S en un alphabet légèrement plus grand $\overline{S} = \{a,b,c,d,\{{b \atop c}\},\{{b \atop d}\},\{{c \atop d}\}\}$. On définit un homomorphisme $\overline{\sigma}$ de $\overline{S}^*$ par

$$\sigma : \begin{cases} \overline{S}^* \to \overline{S}^*, \\ a \mapsto a\{{b \atop c}\}a, \\ b \mapsto d, c \mapsto b, d \mapsto c, \end{cases} \tag{10}$$

étendu naturellement aux sous-ensembles. La raison de cette extension de S à $\overline{S}$ est qu'en plus de l'endomorphisme σ décrit à la section 6.2, il existe un endomorphisme σ' de $\mathfrak{G}$ donné par

$$\sigma'(a) = aba, \quad \sigma'(b) = d, \sigma'(c) = b, \sigma'(d) = c.$$

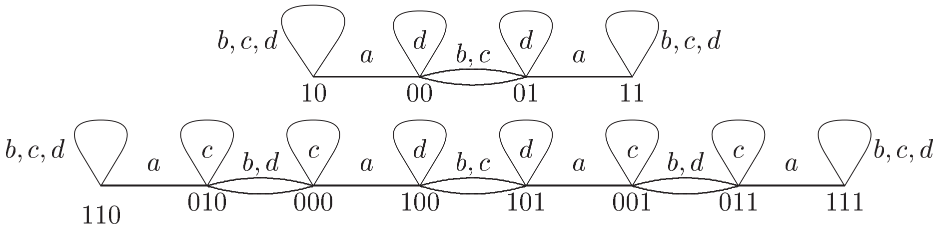
L'opération $\overline{\sigma}$ combine en quelque sorte σ et σ'.

On peut alors vérifier [**Lys85, Bar00d**] que si $w \in S^*$ représente une élément g de $\mathfrak{G}$, alors $\overline{\sigma}(w)$ représente un ensemble d'éléments h de $H = \mathsf{Stab}_{\mathfrak{G}}(1)$ en choisissant dans les lettres-ensemble un élément quelconque, et $\psi(h) = (*, g)$ pour un certain $* \in \mathfrak{G}$.

Soit w le mot $\overline{\sigma}^n(a)$, de longueur $2^n - 1$. On considère à présent le graphe $\mathfrak{G}_n$ suivant : il a 2^n sommets $\{0, 1, \dots, 2^n - 1\}$ alignés sur une droite. L'arête entre $i-1$ et i est étiquetée par w_i (si w_i est un ensemble, il y a 2 arêtes entre $i-1$ et i). On ajoute finalement des boucles en tous les i étiquetées par toutes les lettres de S n'apparaissant pas encore sur une arête en i.

THEOREM 6.6 ([**BG99b**]). $\mathfrak{G}_n \cong \mathcal{S}(\mathfrak{G}, P, S)$.

À titre d'illustration, voici $\mathfrak{G}_2$ et $\mathfrak{G}_3$. On a numéroté les sommets en base 2 pour mieux faire apparaître la structure récursive des graphes :



6.4. Algèbres de Lie

Commençons par quelques généralités sur les suites centrales et les algèbres qu'on peut leur associer. La construction ci-dessous est due à Wilhelm Magnus [**Mag40**] ; elle est par exemple décrite dans [**HB82**, chapitre VIII].

Definition 6.7. Soit G un groupe. Une suite $\Sigma = \{G_n\}_{n=1}^{\infty}$ de sous-groupes de G est une *N-suite* si $G_1 = G$, $G_{n+1} \leq G_n$ et $[G_m, G_n] \leq G_{m+n}$ pour tous les $m, n \geq 1$.

Soit p un nombre premier. Σ est une *p-suite* si de plus $G_n^p \leq G_{pn}$ pour tout n, où G_n^p est le sous-groupe de G_n engendré par ses puissances p-ièmes.

Soit Σ une N-suite. On lui associe l'anneau de Lie gradué

$$\mathcal{L}^{\Sigma}(G) = \bigoplus_{n=1}^{\infty} L_n, \tag{11}$$

où $L_n = G_n/G_{n+1}$ est un groupe abélien, et l'opération de crochet dans $\mathcal{L}^{\Sigma}(G)$ est induite par la commutation dans G. Si de plus Σ est une p-suite, on obtient ainsi une p-algèbre de Lie (ou algèbre restreinte ; voir [**Jac41**] ou [**Jac62**, chapitre V]) sur $\mathbb{F}_p$, dont l'opération de Frobénius $L_n \to L_{pn}$ est induite par la mise à la puissance p.

Comme exemple de N-suite, on peut prendre la suite centrale descendante $\{\gamma_n(G)\}_{n=1}^{\infty}$, donnée par $\gamma_1(G) = G$ et $\gamma_{n+1}(G) = [\gamma_n(G), G]$. La *suite p-centrale descendante*, définie par $P_1(G) = G$ et $P_{n+1}(G) = P_n(G)^p[P_n(G), G]$ donne, elle, une p-suite. De façon générale, si $\Bbbk$ est un corps, on a les *sous-groupes $\Bbbk$-dimensionnels* $\{G_n\}_{n=1}^{\infty}$ définis par

$$G_n = \{g \in G | \, g - 1 \in \Delta^n\}, \qquad n = 1, 2, \ldots$$

où Δ est l'idéal d'augmentation (aussi dit l'idéal fondamental) de l'algèbre de groupe $\Bbbk[G]$. Un résultat de Stephen Jennings [**Jen41**] est que si $\Bbbk$ est de caractéristique p alors $G_n = P_n(G)$.

Si $\mathcal{L}^{\Sigma}(G)$ est un anneau de Lie construit comme en (11) et $\Bbbk$ est un corps, on peut obtenir une algèbre de Lie $\mathcal{L}_{\Bbbk}^{\Sigma}(G)$ sur $\Bbbk$ par

$$\mathcal{L}_{\Bbbk}^{\Sigma}(G) = \bigoplus_{n=1}^{\infty} L_n \otimes_{\mathbb{Z}} \Bbbk.$$

si Σ est la suite $\Bbbk$-dimensionnelle, on écrit simplement $\mathcal{L}_{\Bbbk}(G)$ pour l'algèbre de Lie correspondante.

De nombreuses propriétés d'un groupe transparaissent dans son algèbre de Lie. Par exemple, un des résultats les plus importants obtenus par la méthode décrite ci-dessus est le théorème d'Efim Zel'manov [**Zel95a**] affirmant que si l'algèbre de Lie $\mathcal{L}_{\mathbb{F}_p}(G)$ associée aux sous-groupes $\mathbb{F}_p$-dimensionnels d'un groupe de type fini G résiduellement p et de torsion satisfait une identité polynômiale, alors G est un groupe fini. Il a obtenu ainsi une réponse positive au problème de Burnside restreint [**VZ93, Zel95b, VZ96, Zel97**], qui demande si un groupe de type fini satisfaisant pour un n fixé une équation $X^n = 1$ peut être infini.

Comme autre exemple, mentionnons le critère d'analycité pour les pro-p-groupes, dû à Michel Lazard [**Laz65**]. Il dit qu'un pro-p-groupe est p-adique-analytique (c'est-à-dire qu'il admet une structure de variété analytique sur $\mathbb{Q}_p$ pour laquelle la multiplication est une application analytique) si et seulement si $G_n = G_{n+1}$ pour un n suffisamment grand, où les G_n sont les sous-groupes $\mathbb{F}_p$-dimensionnels de G.

La méthode des algèbres de Lie s'applique aussi à la théorie de la croissance des groupes, comme Rostislav Grigorchuk l'a constaté dans [**Gri89**]. Il a prouvé là que dans la classe des groupes résiduellement p il y a un ń trou ż entre croissance de type polynômial et croissance de type $e^{\sqrt{n}}$. Ce résultat a ensuite été généralisé dans [**LM91**, Theorem D] à la classe des groupes résiduellement nilpotents.

Par ailleurs, Rostislav Grigorchuk et Tullio Ceccherini-Silberstein ont utilisé dans [**CG97**] la méthode des algèbres de Lie pour montrer que de nombreux groupes à un relateur ont croissance uniformément exponentielle, au sens suivant :

Definition 6.8. Soit $\omega_G^S = \lim_{n\to\infty} \sqrt[n]{\gamma_G^S(n)}$ le taux de croissance exponentielle de G relativement au système de générateurs S, et soit $\omega_G = \inf_S \omega_G^S$, où l'infimum est pris sur tous les systèmes de générateurs finis.

Le groupe G a croissance *uniformément exponentielle* si $\omega_G > 1$.

Un lemme classique [**Gri89**] établit le rapport entre la croissance de G et celle de son algèbre de Lie :

Proposition 6.9. *Soit G un groupe engendré par l'ensemble fini S, soit Δ l'idéal d'augmentation de l'algèbre de groupe $\mathbb{F}_pG$, et soit $A = \bigoplus_{n\geq 0} \Delta^n/\Delta^{n+1}$ l'algèbre graduée associée. Alors le taux de croissance ω_G^S de G est au moins égal au taux de croissance de A.*

De plus, A est la p-algèbre enveloppante de $\mathcal{L}(G)$, et donc A et $\mathcal{L}(G)$ ont le même taux de croissance, par l'isomorphisme de Poincaré-Birkhoff-Witt.

Finalement, il y a une tentative de classification de groupes juste-infinis. Rappelons qu'un groupe est *juste-infini* s'il est infini, mais que tous ses quotients propres sont finis. Un groupe G est de *largeur finie* s'il existe une borne $K \in \mathbb{N}$ telle que toutes les inclusions $\gamma_{n+1}(G) < \gamma_n(G)$ soient d'indice au plus K. L'énoncé ci-dessous a été discuté par de nombreux mathématiciens et a été avancé par Efim Zel'manov à Castelvecchio en 1996 [**Zel96**] :

Conjecture 6.10. *Soit G un pro-p-groupe juste-infini de largeur finie. Alors G est soit résoluble, soit p-adique-analytique, soit commensurable à la partie positive d'un groupe de boucles ou au groupe de Nottingham.*

Les calculs de Rostislav Grigorchuk et l'auteur dans [**BG00a**] infirment cette conjecture en produisant un contre-exemple, qui est précisément la complétion profinie du groupe de Grigorchuk $\mathfrak{G}$. Ils obtiennent la structure explicite de $\mathcal{L}_{\mathbb{F}_2}^{\Sigma}(\mathfrak{G})$, pour Σ la suite centrale descendante et la suite 2-centrale descendante.

Le groupe $\mathfrak{G}$ agit sur l'arbre $\mathcal{T}_2$, donc pour tout $m \in \mathbb{N}$ sur $\{0,1\}^m$, qui représente les sommets à distance m de la racine. Soit V_m l'espace vectoriel sur $\mathbb{F}_2$ engendré par $\mathfrak{G}/\operatorname{Stab}_G(0^m)$, de dimension 2^m. Il a une structure de $\mathfrak{G}$-module naturelle. On définit la suite V_m^i de sous-modules de V_m par $V_m^0 = V_m$ et

$$V_m^{r+1} = [\mathfrak{G}, V_m^r] = \langle g\cdot v - v |\, g \in \mathfrak{G}, v \in V_m^r\rangle.$$

Lemma 6.11. *Pour tout $r \in \{0, \dots, 2^m - 1\}$, on a $\dim V_m^r = 2^m - r$.*

On a défini à la section 6.1 les sous-groupes K_m, T_m et N_m. On a les résultats suivants :

Lemma 6.12. *L'application*

$$\alpha \oplus \beta : N_m/N_{m+1} \longrightarrow V_m \oplus V_{m-1}$$

est un isomorphisme de $\mathfrak{G}$-modules pour tout m, où α envoie $(1,\dots,1,x,1,\dots,1) \in K_m$ sur le vecteur de base de V_m correspondant à la position du x, et β envoie $(1,\dots,1,x^2,1,\dots,1) \in T_{m-1}$ sur le vecteur de base correspondant dans V_{m-1}.

Le résultat suivant avait été en partie obtenu par Alexander Rozhkov dans [**Roz96a**].

THEOREM 6.13. *Pour tout $m \geq 1$ on a :*

(1) $\gamma_{2^m+1}(\mathfrak{G}) = N_m$.

(2) $\gamma_{2^m+1+r}(\mathfrak{G}) = N_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_{m-1}^r)$ *for* $r = 0, \ldots, 2^m$.

(3)

$$\dim_{\mathbb{F}_2}(\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})) = \begin{cases} 3 & si\ n = 1, \\ 2 & si\ n = 2, \\ 2 & si\ n = 2^m + 1 + r,\ \ avec\ m > 0\ \ et\ 0 \leq r < 2^{m-1}, \\ 1 & si\ n = 2^m + 1 + r,\ \ avec\ m > 0\ \ et\ 2^{m-1} \leq r \leq 2^m. \end{cases}$$

THEOREM 6.14. *Pour tout $m \geq 1$ on a :*

(1) $\mathfrak{G}_{2^m+1} = N_m$.

(2)

$$\mathfrak{G}_{2^m+1+r} = \begin{cases} N_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_{m-1}^{r/2}) & si\ 0 \leq r \leq 2^m\ \ est\ pair, \\ N_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_{m-1}^{(r-1)/2}) & si\ 0 \leq r \leq 2^m\ \ est\ impair. \end{cases}$$

(3)

$$\dim_{\mathbb{F}_2}(\mathfrak{G}_i/\mathfrak{G}_{i+1}) = \begin{cases} 3 & si\ i = 1, \\ 2 & si\ i > 1\ est\ pair, \\ 1 & si\ i > 1\ est\ impair. \end{cases}$$

Les auteurs introduisent également dans [**BG00a**] la notion de *graphe de Cayley* d'une algèbre de Lie graduée.

DEFINITION 6.15. Soit $L = \bigoplus_{n=1}^{\infty} L_n$ une algèbre de Lie graduée engendrée par un ensemble fini S d'éléments de degré 1. Soit $(\ell_{n,1}, \ldots, \ell_{n,\dim L_n})$ une base de L_n pour tout n, et soit $\langle | \rangle$ le produit scalaire orthonormal sur L associé à cette base. Le *graphe de Cayley* de L est un graphe étiqueté et pondéré défini ainsi : ses sommets sont les points $(i,j) \in \mathbb{N}^2$ avec $i \geq 1$ et $1 \leq j \leq \dim L_n$. Pour tout $s \in S$ et $i,j,k \in \mathbb{N}$ on met une arête de (i,j) à $(i+1,k)$ étiquetée par s et de poids $\langle [\ell_{i,j}, s] | \ell_{i+1,k} \rangle$. Par convention on ne représente pas les arêtes de poids 0. Si de plus L est une p-algèbre, on met pour tout $i,j,k \in \mathbb{N}$ une arête de (i,j) à (pi,k) étiquetée $\cdot^p$ et de poids $\langle \ell_{i,j}^p | \ell_{pi,k} \rangle$, où $\cdot^p$ est l'opération de Frobénius.

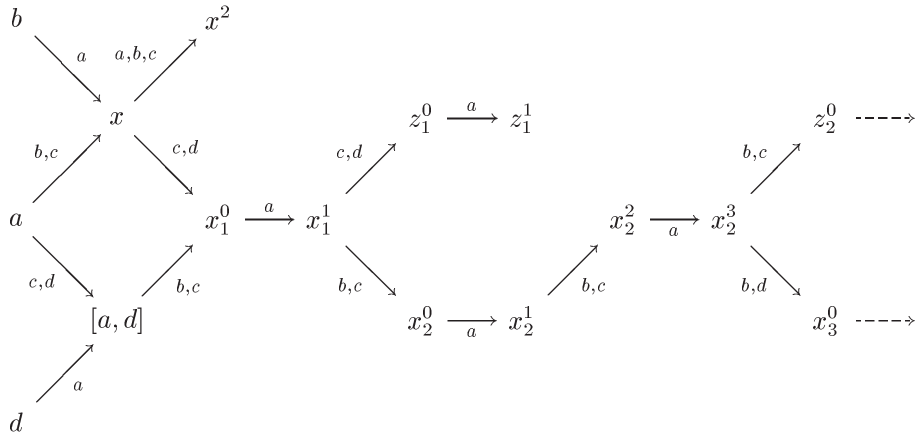
Le graphe de Cayley est une manière géométrique de représenter les constantes de structure de L, et l'algèbre est donc déterminée par son graphe. Ce graphe est connexe, parce que S est un système générateur. La croissance géométrique de graphe de Cayley coïncide avec la série de Hilbert-Poincaré de l'algèbre.

Comme exemple simple, considérons le groupe des quaternions entiers $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, engendré par $\{i, j\}$, et sa suite dimensionnelle $Q_1 = Q$, $Q_2 = \{\pm 1\}$ et $\mathbb{Q}_3 = 1$ sur le corps $\mathbb{F}_2$. Alors le graphe de Cayley de $\mathcal{L}(Q)$ est

$$\begin{tikzcd} i \arrow[dr, "j"] & \\ & -1 \\ j \arrow[ur, "i"'] & \end{tikzcd}$$

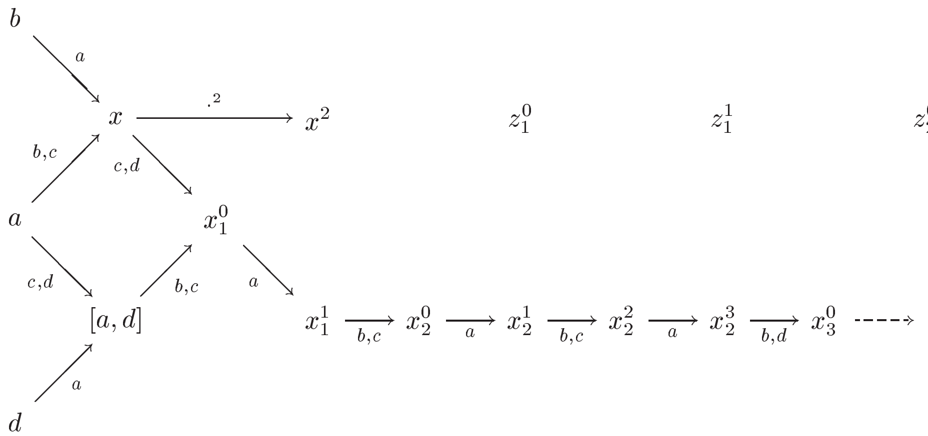
On va maintenant décrire les graphes de Cayley de L et $\mathcal{L}_{\mathbb{F}_2}$ associés respectivement à la suite centrale descendante et à la suite dimensionnelle de $\mathfrak{G}$. Soit $S = \{a, b, c, d\}$ le système générateur de $\mathfrak{G}$, et soit $\overline{S}$ le système de générateurs défini en (10).

THEOREM 6.16. *Le graphe de Cayley de $L(\mathfrak{G})$ est comme suit :*



où $x_m^r = \alpha^{-1}(v_m^r)$ et $z_m^r = \beta^{-1}(v_m^r)$. L'arête $(x_m^{2^m-1}, x_{m+1}^0)$ est étiquetée $\sigma^m\{{c \atop d}\}$, l'arête $(x_m^{2^m-1}, z_m^0)$ est étiquetée $\sigma^m\{{b \atop d}\}$, et les chemins de x_m^0 à $x_m^{2^m-1}$ et de z_m^0 à $z_m^{2^m-1}$ sont étiquetés par $\sigma^{m-1}(a)$.

Le graphe de Cayley de $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$ est comme suit :



avec les mêmes règles pour les étiquettes que pour $L(\mathfrak{G})$; et des flèches étiquetées $\cdot^2$ de x_m^r à z_m^r.

Remarquons que puisque le corps sous-jacent est $\mathbb{F}_2$ tous les poids non-nuls sont égaux à 1 et n'ont donc pas besoin d'être indiqués.

6.5. Croissance

Les résultats principaux sur la croissance de $\mathfrak{G}$ ont été obtenus par Rostislav Grigorchuk lui-même : il montre dans [**Gri83**] que $\mathfrak{G}$ a croissance superpolynômiale et sous-exponentielle, puis dans [**Gri84**] il obtient

$$e^{\sqrt{n}} \precsim \gamma(n) \precsim e^{n^\beta},$$

où $\beta = \log_{32}(31) \approx 0.991$. Cette borne inférieure de $e^{\sqrt{n}}$ est d'ailleurs valable pour tous les groupes non-virtuellement nilpotents (qui, eux, sont de croissance polynômiale — voir [**Wol68, Gro81a**]) qui sont résiduellement p ; voir à la page 48.

Deux de mes articles améliorent chacune de ces bornes. Je montre dans [**Bar98**] le résultat suivant :

THEOREM 6.17.

$$\gamma(n) \precsim e^{n^\alpha},$$

où $\eta \approx 0.811$ *est la racine réelle du polynôme* X^3+X^2+X-2, *et* $\alpha = \log(2)/\log(2/\eta) \approx 0.767$.

La démonstration repose sur le résultat suivant :

LEMMA 6.18. *Soit* $S = \{a,b,c,d\}$, *et soit* $\omega : S \to \mathbb{R}_+^*$ *le poids défini par*

$$\omega(a) = 1-\eta^3 = \eta^2+\eta-1, \qquad \omega(c) = 1-\eta^2,$$
$$\omega(b) = \eta^3 = 2-\eta-\eta^2, \qquad \omega(d) = 1-\eta.$$

Soit $g \in H$, *avec* $\psi(g) = (g_0, g_1)$. *On a alors*

$$\eta(|g|_\omega + |a|_\omega) \geq |g_0|_\omega + |g_1|_\omega.$$

On achève la preuve en montrant que le S-portrait d'un élément $g \in \mathfrak{G}$ contient au plus $|g|_\omega^\alpha$ feuilles, et qu'il n'y a qu'un nombre exponentiel en n de S-portraits à n feuilles.

Ce résultat a été ensuite généralisé à une plus grande famille de groupes par Roman Muchnik et Igor Pak [**MP99**], puis à d'une façon un peu différente par Zoran Šunić et l'auteur dans [**BŠ00**] (voir le chapitre 7).

Une borne inférieure à la fonction de croissance a été obtenue dans [**Bar00d**] : le résultat exact est

THEOREM 6.19.

$$\gamma(n) \succsim e^{n^\alpha},$$

où $\alpha = 0.5157$.

Ce résultat a été obtenu avec l'assistance d'un ordinateur, mais repose sur des considérations purement mathématiques. Le lemme clé est le suivant :

LEMMA 6.20. *Il existe un poids* ω *sur le système de générateurs* $S = \{a,b,c,d\}$ *de* $\mathfrak{G}$, *et des constantes* $K \geq 0$ *et* $\eta \lneqq 4$, *tels que pour tout* $h \in H$, *si on écrit* $\psi(h) = (h_0, h_1)$, *on a*

$$[|h|_\omega \leq \eta \max\{|h_0|_\omega, |h_1|_\omega\} + K. \tag{12}$$

Il suit aisément de ce lemme que la croissance de $\mathfrak{G}$ est au moins e^{n^α}, où $\alpha = \log 2/\log\eta \gneqq 0.5$.

La démonstration du lemme 6.20 consiste en la construction d'un algorithme, exprimé sous la forme d'un transducteur à états finis (apparenté à ceux de la section 1.4) produisant un mot sur S représentant h à partir de mots représentant h_0 et h_1.

Un tel transducteur est malheureusement assez complexe, et un programme informatique a donc été écrit pour le produire. Quelques minutes de calcul ont abouti à un transducteur convenable, possédant 160 états.

Un deuxième programme a ensuite déterminé le poids idéal ω, de façon à minimiser la constante η de l'équation 12. On a obtenu ainsi le poids

$$\omega(a) = 1, \quad \omega(b) = 3.33, \quad \omega(c) = 2.8, \quad \omega(d) = 1.06$$

donnant la constante $\eta \approx 3.84$.

Un résultat légèrement plus faible avait été annoncé par Yuriĭ Leonov [**Leo98a**] : il a obtenu

$$\gamma(n) \succsim e^{n^{0.5041}}$$

par des méthodes assez similaires (mais sans avoir recours à un ordinateur).

CHAPITRE 7

Groupes à colonne vertébrale

Je décris dans ce chapitre un article réalisé en commun avec Zoran Šunić. On a vu plus haut la notion d'automorphismes dirigés à la définition 1.18. On considère ici les groupes suivants :

DÉFINITION 7.1. Un *groupe à colonne vertébrale* agissant sur l'arbre $\mathcal{T}_\Sigma$ est donné par un sous-groupe fini A de $\mathfrak{S}_\Sigma$ agissant transitivement sur le premier niveau Σ de Σ^*, d'un groupe fini B, et d'une suite infinie $\omega = (\omega_1, \omega_2, \dots)$ d'épimorphismes $\omega_i : B \twoheadrightarrow A$. On appelle A le *groupe de racine* et B le *groupe de niveau*.

L'action de $b \in B$ est définie comme suit : on distingue deux lettres de Σ, par exemple 1 et d. On décrète :

$$b(\tau) = \begin{cases} d \dots d1\omega_n(b)(\sigma) & \text{si } \tau \text{ s'écrit } d \dots d1\sigma \text{ avec } n \text{ zéros} \\ \tau & \text{sinon.} \end{cases}$$

Un groupe à colonne vertébrale est *régulier* si A agit régulièrement sur Σ, c'est-à-dire si aucun élément non-trivial de A n'a de point fixe.

Les groupes à colonne vertébrale sont une généralisation des constructions de Rostislav Grigorchuk [**Gri84**] ; ils constituent un revanche un cas particulier des groupes étudiés par Alexander Rozhkov dans [**Roz86**].

Il sort clairement de cette définition que les groupes à colonneăvertébrale sont engendrés par des automorphismes finis et dirigés. Par ailleurs, si on prend $A = \mathbb{Z}/2\mathbb{Z}$ et $B = A \times A$ avec les trois automorphismes u, v, w et la suite $\omega = (u, v, w, u, v, w, \dots)$, on obtient le groupe $\mathfrak{G}$ de la section 2.1.

Considérons une suite ω comme ci-dessus. On note $K_i < B$ le noyau de ω_i, et on note σ le décalage sur les suites :

$$\sigma(\omega_1\omega_2 \dots) = \omega_2 \dots$$

On dit qu'une suite ω est *complète* si pour tout $b \in B$ on a $\omega_i(b) = 1$ pour un $i \in \mathbb{N}$. On impose maintenant sur ω les conditions suivantes :
- pour tout $n \in \mathbb{N}$, la réunion des K_i $(i \geq n)$ est égale à B ;
- pour tout $n \in \mathbb{N}$, l'intersection des K_i $(i \geq n)$ est triviale ;
- pour tout $n \in \mathbb{N}$, la suite $\sigma^n(\omega)$ est complète.

On note $\widehat{\Omega}$ l'ensemble des telles suites. Il est clair que $\widehat{\Omega}$ est invariant par σ.

Les résultats principaux qu'on obtient concernent la croissance des mots et la croissance des périodes. Rappelons cette dernière notion :

DEFINITION 7.2. Soit G un groupe engendré par un ensemble fini S, avec un poids $\omega : S \to \mathbb{R}_+^*$. Sa *fonction de croissance des périodes* est $\pi_\omega : \mathbb{R} \to \mathbb{N}$ définie par

$$\pi_\omega(n) = \max\{\text{ordre}(g) | \, |g|_\omega \leq n\}.$$

On peut mettre sur les fonctions de croissance des périodes la même notion d'équivalence que sur les fonctions de croissance des mots (définie en 3.1). On obtient ainsi un objet intrinsèque à G, c'est-à-dire indépendant de S et ω.

Le théorème suivant étend à une plus grande classe les résultats de [**Gri85**] :

THEOREM 7.3 ([**BŠ00**]). *Soit G_ω un groupe à colonne vertébrale. Alors G_ω a croissance intermédiaire et est de torsion.*

Sa croissance $\gamma(n)$ est au moins e^{n^α}, avec $\alpha = \frac{\log(d)}{\log(d)-\log\frac{1}{2}}$. Si de plus G_ω est régulier, alors sa croissance des périodes $\pi(n)$ est au moins $\sqrt{n}$.

On obtient des résultats plus précis si on suppose qu'il y a de plus une certaine homogénéité dans le mot ω, au sens suivant :

DEFINITION 7.4. La suite ω est *r-homogène* si pour tout $b \in B$ toute sous-suite de longueur r de ω contient un ω_i avec $b \in K_i$.

La suite ω est *r-factorisable* si elle peut s'exprimer comme un produit de suites $\omega'\omega'' \dots$, chacune de longueur au plus r et complète.

Clairement, toute suite r-homogène est r-factorisable, et toute suite r-factorisable est $2r-1$-homogène.

THEOREM 7.5. *Soit ω une suite r-homogène. Alors la fonction de croissance γ de G_ω satisfait*

$$\gamma(n) \precsim e^{n^\beta},$$

où $\beta = \frac{\log(d)}{\log(d)-\log(\eta_r)}$ et η_r est la racine positive du polynôme $X^r + X^{r-1} + X^{r-2} - 2$.

Si de plus G_ω est régulier, alors la fonction de croissance des périodes π de G_ω satisfait

$$\pi(n) \precsim n^{\log_{1/\eta_r}(d)}.$$

On a le résultat suivant pour les suites r-factorisables. Il est intéressant de noter que selon la suite ω choisie l'un ou l'autre des théorèmes 7.5 et 7.6 peut donner la meilleure estimation.

THEOREM 7.6. *Soit ω une suite r-factorisable. Alors la fonction de croissance γ de G_ω satisfait*

$$\gamma(n) \precsim e^{n^\beta},$$

où $\beta = \frac{\log(d)}{\log(d)-\frac{1}{r}\log(4/3)}$.

Si de plus G_ω est régulier, alors la fonction de croissance des périodes π de G_ω satisfait

$$\pi(n) \precsim n^{r\log_{4/3}(d)}.$$

Il existe en fait une relation générale reliant la croissance des mots et la croissance des périodes, dûe à Yuriĭ Leonov [**Leo99**]. Elle est basée sur une étude soigneuse de la notion de portrait. Dans notre cadre, elle se formule comme suit :

THEOREM 7.7. *Soit G_ω un groupe à colonne vertébrale régulier, où ω est une suite r-factorisable. Si on a*

$$\gamma(n) \precsim e^{n^\beta},$$

alors on a aussi

$$\pi(n) \precsim n^{r\beta}.$$

Deuxième partie

Croissance dans des graphes

CHAPITRE 8

Croissance des chemins

Dans cette partie, je rappelle les résultats obtenus dans [**Bar99**]. Ils concernent à proprement parler des digraphes quelconques, mais trouvent leur application principale en théorie des groupes, dans la célèbre ń formule de Grigorchuk ż (15).

Definition 8.1 (Graphes). Un *graphe* $\mathfrak{G}$ est une paire d'ensembles $\mathfrak{G} = (V, E)$ (les *sommets* et *arêtes* ; notés ainsi à cause de leurs noms anglais ń vertices ż et ń edges ż) munie d'applications

$$\alpha : E \to V, \qquad \omega : E \to V, \qquad \bar{\cdot} : E \to E$$

satisfaisant

$$\overline{\overline{e}} = e, \qquad \alpha(\overline{e}) = \omega(e).$$

Le graphe $\mathfrak{G}$ est dit *fini* si $E(\mathfrak{G})$ et $V(\mathfrak{G})$ sont les deux des ensembles finis.

Un *morphisme de graphes* $\phi : \mathfrak{G} \to \mathfrak{H}$ est une paire d'applications $(V(\phi), E(\phi))$ avec $V(\phi) : V(\mathfrak{G}) \to V(\mathfrak{H})$ et $E(\phi) : E(\mathfrak{G}) \to E(\mathfrak{H})$ telles que

$$\overline{E(\phi)e} = E(\phi)\overline{e}, \qquad V(\phi)(\alpha e) = \alpha(E(\phi)e).$$

Étant donné une arête $e \in E(\mathfrak{G})$, on appelle $\alpha(e)$ et $\omega(e)$ respectivement la *source* et la *destination* de e. Deux sommets x, y sont *adjacents*, ce qu'on note $x \sim y$, s'ils sont reliés par une arête, c'est-à-dire s'il existe une arête $e \in E(\mathfrak{G})$ telle que $\alpha(e) = x$ et $\omega(e) = y$. Deux arêtes e, f sont *consécutives* si $\omega(e) = \alpha(f)$. Une *boucle* est une arête e avec $\alpha(e) = \omega(e)$.

Le *degré* $\deg(x)$ d'un sommet x est le nombre d'arêtes incidentes :

$$\deg(x) = \#\{e \in E(\mathfrak{G}) | \, \alpha(e) = x\} = \#\{e \in E(\mathfrak{G}) | \, \omega(e) = x\}.$$

Si $\deg(x)$ est fini pour tout sommet x, on dit que $\mathfrak{G}$ est *localement fini*. Si $\deg(x) = d$ pour tout sommet x, on dit que $\mathfrak{G}$ est *d-régulier*.

Notons que l'involution $e \mapsto \overline{e}$ peut avoir des points fixes ; ce seraient des boucles ń égales à elles-mêmes parcourues en sens inverse ż, ce qui ne se représente pas aisément. Au cas où $\bar{\cdot}$ n'a pas de point fixe, $\mathfrak{G}$ peut être vu comme un graphe simplicial — il est représenté fidèlement par un dessin dans $\mathbb{R}^3$.

Cette notion de graphe est particulièrement bien adaptée à la théorie des groupes. Les graphes rencontrés habituellement en combinatoire sont donnés par un ensemble V et une famille E de sous-ensembles de cardinalité 2 de V. Dans notre contexte, cela revient à interdire les points fixes de $\bar{\cdot}$; à exiger de toute arête e que $\alpha(e)$ et $\omega(e)$ soient distincts ; et, pour toute paire de sommets x, y, à autoriser au plus une arête entre x et y. Aucune de ces restrictions n'est nécessaire pour notre propos.

Bien que chaque arête de $\mathfrak{G}$ soit individuellement orientée, le graphe $\mathfrak{G}$ lui-même doit être considéré comme un graphe non-orienté, vu que chaque arête e possède une image-miroir $\overline{e}$.

Definition 8.2 (Chemins). Un *chemin* dans $\mathfrak{G}$ est une séquence π,

$$\pi = (v_0, e_1, v_1, e_2, \ldots, e_n, v_n)$$

de sommets et d'arêtes de $\mathfrak{G}$, avec $\alpha(e_i) = v_{i-1}$ et $\omega(e_i) = v_i$ pour tout $i \in \{1, \ldots, n\}$, avec $n \geq 0$. La *longueur* du chemin π est le nombre n d'arêtes dans π. Le *début* du chemin π est $\alpha(\pi) = v_0$, et sa *fin* est $\omega(\pi) = v_n$. Si $\alpha(\pi) = \omega(\pi)$, le chemin π est appelé un *circuit* en $\alpha(\pi)$. Dans la plupart des cas, on peut omettre les v_n de la description des chemins ; ils ne sont nécessaires que quand $|\pi| = 0$, auquel cas un point de départ doit être fixé. On étend l'involution $\bar{\ }$ des arêtes aux chemins par

$$\overline{\pi} = (v_n, \overline{e_n}, \ldots, v_1, \overline{e_1}, v_0)$$

(notons que $\overline{\pi}$ est un chemin de $\omega(\pi)$ à $\alpha(\pi)$).

On note $E^*(\mathfrak{G})$ l'ensemble des chemins, avec une multiplication partiellement définie par la concaténation des suites : si π et ρ sont doux chemins avec $\omega(\pi) = \alpha(\rho)$, leur *produit* est défini par $\pi\rho = (\pi_1, \ldots, \pi_{|\pi|}, \rho_1, \ldots, \rho_{|\rho|})$. Étant donnés deux sommets $x, y \in V(\mathfrak{G})$, on note $[x, y]$ l'ensemble des chemins de x à y. On fait de $V(\mathfrak{G})$ un espace métrique en définissant pour des sommets $x, y \in V(\mathfrak{G})$ leur *distance*

$$d(x, y) = \min_{\pi \in [x,y]} |\pi|.$$

La multiplication des chemins est une opération $[x, y] \times [y, z] \to [x, z]$. Cela fait de $\mathfrak{G}$ une catégorie libre, avec pour objets $V(\mathfrak{G})$ et pour morphismes $\mathrm{Hom}(x, y) = [x, y]$.

DEFINITION 8.3 (Bosses). Le chemin π a une *bosse en i* si $\pi_i = \overline{\pi_{i+1}}$; si la position de la bosse n'est pas importante, on dit juste que π a une bosse. Le *nombre de bosses* $\mathrm{nb}(\pi)$ d'un chemin est le nombre d'indices i où π a une bosse. Un chemin est /chemin !propre]propre s'il n'a pas de bosses.

On supposera désormais qu'un graphe localement fini $\mathfrak{G}$ a été fixé, avec deux sommets distingués $\star, \dagger \in V(\mathfrak{G})$. On souhaite énumérer les chemins, en comptant leur nombre de bosses, entre $\star$ et $\dagger$ dans $\mathfrak{G}$. On définit :

DEFINITION 8.4 (Séries de chemins). La série formelle

$$G(t) = \sum_{\pi \in [\star, \dagger]} t^{|\pi|} \in \mathbb{N}[[t]]$$

est appelée la *série de chemins* de $(\mathfrak{G}, \star, \dagger)$. La série

$$F(u, t) = \sum_{\pi \in [\star, \dagger]} u^{\mathrm{nb}(\pi)} t^{|\pi|} \in \mathbb{N}[u][[t]] \subset \mathbb{N}[[u, t]]$$

est appelée la *série de chemins enrichie* de $(\mathfrak{G}, \star, \dagger)$. Sa spécialisation $F(0, t)$ est appelée la *série de chemins propres* de $(\mathfrak{G}, \star, \dagger)$.

Au cas où $\star = \dagger$, on appelle G la *série de circuits* de $(\mathfrak{G}, \star)$ et F la *série de circuits enrichie* de $(\mathfrak{G}, \star)$. La série de circuits est souvent appelée la /fonction(s) !de Green]fonction de Green du graphe $\mathfrak{G}$; puisqu'on s'intéresse plus ici à des propriétés formelles de G, on l'appellera aussi la *série de Green* de $\mathfrak{G}$.

Notons que $G(t)$ appartient à $\mathbb{N}[[t]]$ parce que $\mathfrak{G}$ est localement fini, donc il n'y a qu'un nombre fini de chemins de longueur donnée, et ainsi qu'un nombre fini de termes dans la somme qui ont un degré donné.

De même, $F(u, t)$ appartient à $\mathbb{N}[u][[t]]$, parce que si on exprime $F(u, t) = \sum f_n t^n$ avec à priori $f_n \in \mathbb{N}[[u]]$, chaque f_n n'est qu'une somme finie de monômes, donc est un polynôme. Si on pose $G(t) = \sum g_n t^n$, on a même $g_n = f_n(1)$.

Le résultat principal de [**Bar99**] est, pour des graphes d-réguliers, le

THEOREM 8.5. *Soit $\mathfrak{G}$ un graphe d-régulier. On a alors*

$$\frac{F(1-u, t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}. \tag{13}$$

CHAPITRE 9

Applications aux groupes

Montrons maintenant en quoi F est relié à la cocroissance et G aux marches aléatoires dans des groupes. On généralisera ainsi la formule de Grigorchuk (valable pour des quotients de groupes libres) à des espaces homogènes de groupes libres. Pour un survol de la théorie des marches aléatoires sur des groupes, voir par exemple les références [**MW89, Woe94**].

Dans ce chapitre on écrira toujours $F(t)$ pour $F(0,t)$. Rappelons la notion de croissance dans les groupes :

Definition 9.1. Soit Γ un groupe engendré par un système fini symétrique S. Pour $\gamma \in \Gamma$, on définit sa *longueur*

$$|\gamma| = \min\{n \in \mathbb{N} : \gamma \in S^n\}.$$

La *série de croissance* de (Γ, S) est la série formelle

$$f_{(\Gamma,S)}(t) = \sum_{\gamma \in \Gamma} t^{|\gamma|} \in \mathbb{N}[[t]].$$

En développant $f_{(\Gamma,S)}(t) = \sum f_n t^n$, la *croissance* de (Γ, S) est

$$\alpha(\Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n}$$

(cette limite supérieure est en fait une limite et est bornée par $|S|-1$).

Soit R un sous-ensemble de Γ. La *série de croissance* de R par rapport à (Γ, S) est la série formelle

$$f^R_{(\Gamma,S)}(t) = \sum_{\gamma \in R} t^{|\gamma|} \in \mathbb{N}[[t]].$$

En développant $f^R_{(\Gamma,S)}(t) = \sum f_n t^n$, on définit la *croissance* de R par rapport à (Γ, S) comme

$$\alpha(R; \Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n}.$$

Si X est un G-ensemble transitif, la *marche aléatoire simple* sur (X, S) est la marche aléatoire d'un point sur X, ayant probabilité $1/|S|$ de se déplacer de sa position courante x à un voisin $s \cdot x$, pour tout $s \in S$. Fixons un point $\star \in X$, et notons p_n la probabilité qu'une marche aléatoire commençant en $\star$ arrive à $\star$ après n déplacements. On définit le *rayon spectral* de la marche aléatoire comme

$$\nu(X, S) = \limsup_{n\to\infty} \sqrt[n]{p_n}.$$

Ce nombre est indépendant du choix de $\star$.

La *série de cocroissance* (respectivement la *cocroissance*) de (Γ, S) est définie comme la série de croissance (respectivement la croissance) de $\ker(\pi : \Pi \to \Gamma)$ par rapport à $(\mathbb{F}_S, S)$, où $\mathbb{F}_S$ est le groupe libre sur S.

On peut associer à un groupe Π engendré par un ensemble symétrique S et un sous-groupe Ξ un graphe $|S|$-régulier $\mathfrak{G}$ sur lequel Π agit transitivement. On l'appelle le *graphe de Schreier* de (Π, S) par rapport à Ξ. Il est défini comme $\mathfrak{G} = (E, V)$, où

$$V = \Pi/\Xi$$

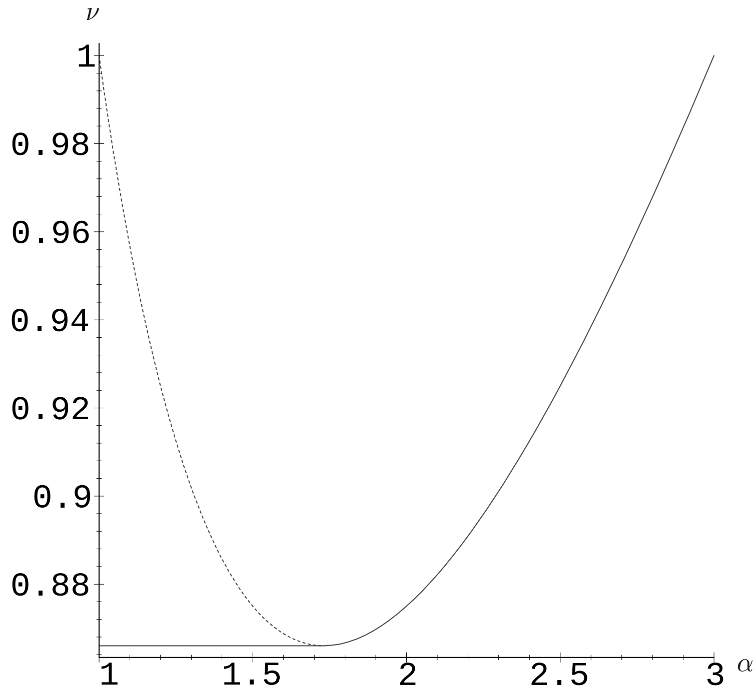


FIG. 9.1 – La correspondance $\alpha \mapsto \nu$ reliant la cocroissance et le rayon spectral (si $|S| = 4$)

et

$$E = V \times S, \quad \alpha(v,s) = v, \quad \omega(v,s) = sv, \quad \overline{(v,s)} = (sv, s^{-1});$$

c'est-à-dire que deux classes à gauche A, B sont reliées par au moins une arête si et seulement si $SA \supset B$. (On retrouve le graphe de Cayley de (Π, S) si $\Xi = 1$). Il y a une boucle en $v\Xi$ dans $\mathfrak{G}$ chaque fois que $v\Xi v^{-1} \cap S \neq \emptyset$; et il y a $|v\Xi w^{-1} \cap S|$ arêtes entre $v\Xi$ et $w\Xi$ dans $\mathfrak{G}$.

En identifiant F à la croissance relative de Ξ et G à la fonction de Green de $\mathbb{F}_S/\Xi$, on obtient le corollaire suivant :

COROLLARY 9.2. *Soit Ξ un sous-groupe de $\mathbb{F}_S$. Soit $\nu = \nu(\Xi \backslash \Pi, S)$ le rayon spectral de la marche aléatoire simple sur $\mathbb{F}_S/\Xi$, et soit $\alpha = \alpha(\Xi; \Pi, S)$ la croissance relative de Ξ dans Π. On a alors*

$$\nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|}\left(\frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha}\right) & \text{si } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{si } \alpha \leq \sqrt{|S|-1}. \end{cases} \tag{14}$$

En se restreignant au cas où Ξ est normal, avec $\mathbb{F}_S/\Xi = \Gamma$, on obtient le résultat suivant :

COROLLARY 9.3 (Grigorchuk [**Gri80b**]). *Soit Γ un groupe engendré par un ensemble symétrique fini S, soit ν le rayon spectral de la marche aléatoire simple sur Γ, et soit α*

la cocroissance de (Γ, S). *Alors*

$$\nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|}\left(\frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha}\right) & \text{si } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{sinon.} \end{cases} \tag{15}$$

Une grande quantité de démonstrations existent pour ce résultat : la preuve originale [**Gri80b**] due à Rostislav Grigorchuk, une preuve de Cohen [**Coh82**], une extension par Northshield à des graphes transitifs [**Nor92**], une preuve courte due à Szwarc [**Szw89**] utilisant la théorie des algèbres d'opérateurs, une autre de Woess [**Woe94**], etc.

On remarque en fait que si $\alpha < \sqrt{|S|-1}$, alors nécessairement $\alpha = 0$. De façon équivalente, si $\alpha < \sqrt{|S|-1}$, alors $\Xi = 1$, et le graphe de Cayley $\mathfrak{G}$ est un arbre. Il est même connu que si $\alpha \neq 0$ alors $\alpha > \sqrt{|S|-1}$; voir [**Pas93**] pour cette petite amélioration.

Dans [**Bar99**] une formule analogue à (15) est donnée pour des groupes Γ quotients de $PSL_2(\mathbb{Z})$; les séries de Green et de cocroissance sont définies relativement à $PSL_2(\mathbb{Z})$ engendré par les matrices $\left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ et $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, et et non au groupe libre $\mathbb{F}_S$ engendré par l'ensemble de générateurs S.

CHAPITRE 10

Calculs pour des arbres

Soit $\mathfrak{G}$ un arbre d-régulier. On va calculer pour $\mathfrak{G}$ les séries F et G introduites au chapitre 8, reproduisant ainsi des calculs faits par Harry Kesten [**Kes59**].

D'abord, il est clair que $F(0,t) = 1$. En effet, un arbre n'a par définition pas de circuit, et donc tout chemin fermé non-trivial dans $\mathfrak{G}$ contient nécessairement une bosse. Par conséquent, le seul monôme dans $F(u,t)$ de u-degré 0 est le monôme 1.

Fixons une origine $\star$ de $\mathfrak{G}$, et ôtons à $\mathfrak{G}$ une arête e incidente à $\star$. Appelons $\mathcal{T}$ la composante connexe de $\mathfrak{G} \setminus \{e\}$ contenant $\star$. Par exemple, si $d = 2$, on a $\mathfrak{G} \cong \mathbb{Z}$ et $\mathcal{T} \cong \mathbb{N}$. Soient respectivement $F(u,t)$ et $T(u,t)$ les séries comptant les circuits dans $\mathfrak{G}$ et dans $\mathcal{T}$. On a alors les équations suivantes :

$$T = 1 + (d-1)t(T-1+u)t\frac{1}{1-(d-2+u)t(T-1+u)t},$$
$$F = 1 + dt(T-1+u)t\frac{1}{1-(d-1+u)t(T-1+u)t}.$$

Expliquons ces formules. Un circuit dans $\mathcal{T}$ est soit le circuit trivial (contribuant 1), soit une suite de circuits dans des graphes isomorphes à $\mathcal{T}$. Cette suite est faite, d'abord, d'un pas dans une des $d-1$ directions issues de $\star$, puis d'un ń sous-circuit ż ne revenant pas à $\star$ en ce point, puis d'un pas pour revenir à $\star$; puis d'un nouveau pas dans une des $d-1$ directions issues de $\star$, etc. Si le sous-circuit est trivial, une bosse apparaît dans le circuit, et si la nouvelle direction de sortie est la même que la précédente, une bosse apparaît aussi.

De même, un circuit dans $\mathfrak{G}$ peut s'exprimer à l'aide d'une suite de circuits dans des sous-graphes isomorphes à $\mathcal{T}$.

Ces équations peuvent aisément être résolues, donnant

$$T(1-u,t) = \frac{2(1-u)}{1-u(d-u)t^2+\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}},$$
$$F(1-u,t) = \frac{2(d-1)(1-u^2t^2)}{(d-2)(1+u(d-u)t^2)+d\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}}.$$

Mais maintenant, on constate qu'en utilisant l'équation (13, page 58) et $F(0,t) = 1$ on obtient

$$G(t) = \frac{1+(d-1)\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2}{1-\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2},$$

ou, après simplification,

$$G(t) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}},$$

qui permet aussi, par une deuxième utilisation de (13), d'obtenir $F(1-u,t)$.

En particulier, si $d = 2$, on a $\mathfrak{G} = C_\infty = \mathbb{Z}$, et

$$G(t) = \sum_{n\geq 0} \binom{2n}{n} t^{2n} = \frac{1}{\sqrt{1-4t^2}}.$$

Remarquons que pour tout d l'arbre d-régulier est le graphe de Cayley de $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{*d}$ avec son système de générateurs standard. Si d est pair, $\mathfrak{G}$ est aussi le graphe de Cayley d'un groupe libre de rang $d/2$ engendré par un système libre de générateurs. On peut ainsi calculer le rayon spectral de la marche aléatoire sur un groupe libre engendré librement : il est, pour $\mathbb{F}_{d/2}$, égal à

$$\frac{2\sqrt{d-1}}{d}. \tag{16}$$

On peut même étudier une série génératrice plus riche que F : soit

$$H(u,v,t) = \sum_{\pi\text{: chemin issu de } \star} t^{|\pi|} u^{\mathrm{nb}(\pi)} v^{d(\star,\pi_{|\pi|})} \in \mathbb{N}[u,v][[t]],$$

où $d(-,-)$ désigne la distance dans $\mathfrak{G}$. On a alors

$$\begin{aligned} H(1,v,t) &= F(1,t) + d(T-1+u)tvF + d(T-1+u)tv(d-1)(T-1+u)tvF + \dots \\ &= \frac{1+(T-1+u)(1,t)tv}{1-(d-1)(T-1+u)(1,t)tv} F(1,t); \end{aligned}$$

et du fait que H est une somme de séries comptant les chemins entre deux points fixés, on peut obtenir $H(u,v,t)$ de $H(1,v,t)$ en étendant (13) linéairement à $\mathbb{N}[u,v][[t]]$:

$$\frac{H(1-u,v,t)}{1-u^2t^2} = \frac{H\left(1, v, \frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}.$$

On aurait aussi pu commencer par calculer la série de croissance

$$H(0,v,t) = \frac{1+vt}{1-(d-1)vt}$$

de tous les chemins propres dans $\mathfrak{G}$, puis inverser l'égalité (13) pour obtenir

$$H(1,v,t) = \frac{1+\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2t}\right)^2}{1-u^2\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2} \cdot H\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}, 0, v\right),$$

$$H(u,v,t) = \frac{1-t^2u^2}{1+u(d-u)t^2} \cdot \frac{(d-1)(4t^2+\square^2)}{4(d-1)^2t^2-u^2\square^2} \cdot \frac{2(d-1)t+v\square}{2t-v\square},$$

où $\square = 1+u(d-u)t^2 - \sqrt{(1+u(d-u)t^2)^2 - 4(d-1)t^2}$.

Souvenons-nous que la série de croissance d'un graphe $\mathfrak{G}$ en un point base $\star$ est la série formelle

$$P(t) = \sum_{v\in V(\mathfrak{G})} t^{d(\star,v)} \in \mathbb{N}[[t]],$$

La série H que nous venons de décrire est assez générale pour contenir les informations suivantes sur $\mathfrak{G}$:

- $H(u,0,t) = F(u,t)$;
- $H(0,1,t) = \frac{1+t}{1-(d-1)t} = P(t)$ est la série de croissance de $\mathcal{T}$;
- $H(1,1,t) = 1/(1-dt)$ est la série de croissance de tous les chemins dans $\mathcal{T}$.

CHAPITRE 11

Séries de produits directs et de produits libres

On donne ici la construction d'un graphe pointé à partir de deux graphes pointés, qui s'apparente ou produit libre dans les groupes ou les catégories. Notre but est d'obtenir des formules exprimant la série de Green du produit en terme des séries de Green des facteurs.

Cette définition est dûe à Gregory Quenell [**Que94**, Definition 4.8] :

Definition 11.1. Soit $(\mathfrak{E}, \star)$ et $(\mathfrak{F}, \star)$ deux graphes pointés. Leur *produit libre* $\mathfrak{E} * \mathfrak{F}$ est le graphe construit comme suit : on commence par prendre une copie de $\mathfrak{E}$ et une de $\mathfrak{F}$, et on les identifie en leur point base $\star$. Maintenant, en tout sommet de $\mathfrak{E}$ (excepté $\star$) on colle une copie de $\mathfrak{F}$ à son point base, et en tout sommet de $\mathfrak{F}$ (excepté $\star$) on colle une copie de $\mathfrak{E}$ à son point base. On procède de même pour les sommets des nouvelles copies.

Si (E, S) et (F, T) sont deux groupes avec systèmes de générateurs fixés, dont les graphes de Cayley sont $\mathfrak{E}$ et $\mathfrak{F}$ respectivement, alors $\mathfrak{E} * \mathfrak{F}$ est le graphe de Cayley de $(E * F, S \sqcup T)$.

L'arbre d-régulier peut aussi être obtenu comme produit libre de d graphes en forme de "I", c'est-à-dire de graphes réguliers de degré 1.

Le résultat obtenu dans [**Bar99**] est :

Theorem 11.2. *Soient $G_{\mathfrak{E}}$ et $G_{\mathfrak{F}}$ les séries de Green de deux graphes $\mathfrak{E}$ et $\mathfrak{F}$. Alors*

$$\frac{1}{(tG_{\mathfrak{E}*\mathfrak{F}})^{-1}} = \frac{1}{(tG_{\mathfrak{E}})^{-1}} + \frac{1}{(tG_{\mathfrak{F}})^{-1}} - \frac{1}{t}, \tag{17}$$

où $F^{-1}(t)$ est l'inverse formel de la série F, c'est-à-dire une série G telle que $G(F(t)) = F(G(t)) = t$.

Une équation équivalente à celle-ci, mais de manière non-triviale, apparaît dans un article de Gregory Quenell [**Que94**], et, dans un langage complètement différent (celui des variables aléatoires non-commutatives), dans un article de Dan Voiculescu [**Voi90**, Theorem 4.5].

Corollary 11.3. *Si les séries de Green de $\mathfrak{E}$ et $\mathfrak{F}$ sont les deux algébriques, alors la série de Green de $\mathfrak{E} * \mathfrak{F}$ est aussi algébrique.*

On peut aussi utiliser (17) pour obtenir d'une troisième manière la série de Green d'un arbre régulier (voir le chapitre 10). En effet, de façon générale, le produit libre d'arbres réguliers de degré d et e est un arbre régulier de degré $d + e$. Soit G_d la série de Green de l'arbre régulier de degré d.

On calcule d'abord $G_1 = \frac{1}{1-t^2}$, par l'observation que tous les chemins dans l'arbre 1-régulier sont des aller-retour le long de l'unique arête.

On sait ensuite par (17) que

$$\frac{1}{(tG_d)^{-1}} = d\frac{1}{(tG_1)^{-1}} - \frac{d-1}{t},$$

d'où on sort que

$$(tG_d)^{-1}(u) = \frac{2u}{2 - d + d\sqrt{1 + 4u^2}},$$

et finalement

$$G_d(t) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}},$$

comme on l'a vu auparavant.

Troisième partie

Croissance des groupes agissant sur des arbres

CHAPITRE 12

Groupes GGS

On définit à présent une famille de groupes à automates, qui généralise une construction due à Rostislav Grigorchuk, Narain Gupta et Saïd Sidki. On donnera des résultats nouveaux généralisant à une famille de groupes agissant sur des arbres certaines propriétés algébriques et de croissance.

DEFINITION 12.1. Soit A un sous-groupe du groupe symétrique $\mathfrak{S}_\Sigma$ agissant transitivement sur l'alphabet Σ de cardinalité d, et soient $\epsilon_1, \dots, \epsilon_{d-1} \in A$ des permutations de Σ engendrant A.

On définit l'automorphisme automatique t_ϵ comme suit : pour une suite $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \Sigma^*$, on pose par induction $t_\epsilon(\emptyset) = \emptyset$ et

$$t_\epsilon(\sigma) = \begin{cases} \phi_{\sigma_1}(\epsilon_{\sigma_1})(\sigma) & \text{si } \sigma_1 \neq d, \\ \sigma_1 t_\epsilon(\sigma_2 \dots \sigma_n) & \text{si } \sigma_1 = d \end{cases}$$

(l'application ϕ_σ a été définie en (3, page 19)).

Le *groupe GGS* associé à A et ϵ est le sous-groupe de $\mathsf{Aut}\,\mathcal{T}$ engendré par A et t_ϵ. On le note G_ϵ, rendant A implicite dans la notation. On appelle A le *groupe de la racine* et ϵ la *suite de récurrence*.

Dans les définitions précédentes (voir par exemple [**Gri00**]), on supposait toujours A cyclique et d premier. On ne met pas ici ces deux conditions ; elles reviennent à exiger que d soit premier et que G soit un sous-groupe du pro-d-Sylow $\mathsf{Aut}_*\,\mathcal{T}$.

Si on ne suppose pas d'emblée que A agit transitivement et est engendré par les ϵ_i, on dira que G est un *groupe proto-GGS*.

On considère donc les groupes engendrés par des automorphismes finis confinés au premier niveau, et un automorphisme dirigé t_ϵ selon le chemin $dd\dots$, au sens de la définition 1.18. On remarque que t_ϵ appartient au stabilisateur du premier niveau $\mathsf{Stab}_G(1)$. On montre à la figure 12.1 un automate correspondant au générateur t_ϵ — voir la définition de son action sur l'arbre $\mathcal{T}_\Sigma$ à la section 1.4.

Quant au portrait de t_ϵ relativement à A, au sens de la définition 1.6, il peut être décrit simplement comme ceci (voir la figure 12.2) : l'étiquette d'un sommet σ de la forme $dd\dots d\sigma_n$, avec $\sigma_n \neq d$, est ϵ_{σ_n}. Toutes les autres étiquettes sont 1.

12.1. Exemples

Il règne une très grande variété parmi les groupes GGS, et nous nous bornerons à considérer les quelques exemples les plus simples. Naturellement, si $d = 1$ il n'y a que l'automorphisme trivial, et on suppose d'emblée $d \geq 2$. Voyons tout d'abord que si $d = 2$ les exemples sont tous bien connus :

PROPOSITION 12.2. *Soit G un groupe GGS agissant sur l'arbre $\mathcal{T}_2$. Alors G est isomorphe à $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, le groupe diédral infini.*

DÉMONSTRATION. On a forcément $A = \mathbb{Z}/2\mathbb{Z}$ engendré par a, et $\epsilon_1 = a$. On constate que t_ϵ est d'ordre 2, et il ne reste qu'à montrer que l'élément $g = at_\epsilon$ est d'ordre infini.

On a $g^2 = \phi_1(g^a)\phi_2(g)$, et donc en itérant $g^{2^n} = \phi_{1^n}(g^a) \cdots \phi_{2^n}(g)$. Tous ces éléments sont distincts, car g^{2^n} appartient à $\mathsf{Stab}_G(n)$ mais pas à $\mathsf{Stab}_G(n+1)$. □

Déjà quand $d = 3$ et $A = \mathfrak{A}_3$, le groupe alterné, on a des exemples très intéressants :

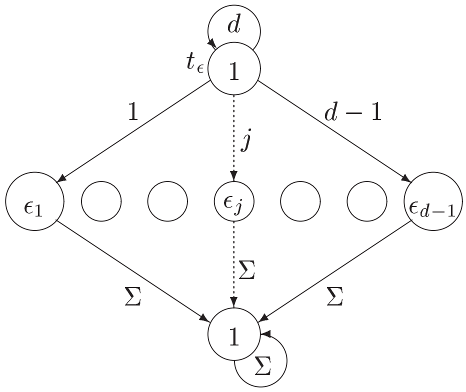


FIG. 12.1 – Un automate agissant comme t_ϵ

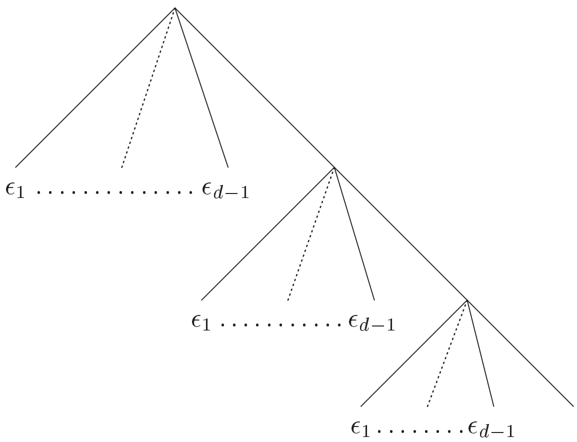


FIG. 12.2 – L'étiquetage du portrait de t_ϵ relativement à A

- $G_{(123),()}$ a été étudié par Jacek Fabrykowski et Narain Gupta [**FG85, FG91**], qui ont étudié son taux de croissance : ils montrent qu'il est de croissance intermédiaire. On le nomme Γ à la section 2.1.
- $G_{(123),(123)}$ a été étudié par l'auteur et Rostislav Grigorchuk dans [**BG99b**] et [**BG99a**]. On le nomme $\overline{\Gamma}$ à la section 2.1.
- $G_{(123),(132)}$ a été étudié par Narain Gupta et Saïd Sidki [**GS83b**], qui ont montré qu'il est de torsion, puis qu'il est de croissance intermédiaire, et ont obtenu des résultats très fins sur sa structure de sous-groupes et ses automorphismes. On le nomme $\overline{\overline{\Gamma}}$ à la section 2.1.

On peut considérer aussi l'exemple suivant, très intéressant : on prend $d = 3$, $A = \mathfrak{S}_3$, et on considère le groupe $G = G_{(12),(23)}$. Quelques résultats faciles, qu'on ne montrera pas, sont : G est un groupe fractal. Il admet des quotients G_n agissant sur Σ^n, qu'on peut décrire ainsi : si on note $v : \mathfrak{S}_3 \to \pm 1$ l'application signature, on peut décrire G_n en termes de portraits : ses éléments ont précisément les portraits Π tels que $v(\Pi(\sigma 1)\Pi(\sigma 2)\Pi(\sigma 3)) =$

1. On a donc
$$|G_n| = 2^{3^{n-1}} 3^{\frac{1}{2}(3^n-1)}.$$
En particulier, les 3-Sylow de G_n et $\mathsf{Aut}\,\mathcal{T}_3/\,\mathsf{Stab}(n)$ sont isomorphes.

Il suit aussi de cette description que la complétion profinie de G est un pro-groupe à branches régulières ; en effet, considérons le sous-groupe K de G engendré par (123) et $t_{(12)(23)}$. Sa clôture $\widehat{K}$ dans $\widehat{G}$ est d'indice 2, et contient $\widehat{K} \times \widehat{K} \times \widehat{K}$ comme sous-groupe d'indice 12. Toutefois, on ne sait pas si G lui-même est à branches, car G n'a pas la propriété de congruence de la définition 1.8.

Pour $d = 4$, trois exemples ont été considérés par Grigorchuk, et toujours avec $A = \langle a = (1234)\rangle$ cyclique. Il s'agit de $G_{a,1,a}$, $G_{a,1,a^3}$ et $G_{1,a,1}$. Il a conjecturé, lors du congrès de Champoussin sur la croissance des groupes en 1994, qu'ils sont de croissance intermédiaire. On répondra à cette question dans le chapitre 14.

12.2. Une première réduction

Deux des hypothèses données dans la définition de groupe GGS — la transitivité de A, et le fait qu'il est engendré par $\{\epsilon_1, \dots, \epsilon_{d-1}\}$ — ne sont pas essentielles.

Dans cette section, on suppose que G est un groupe proto-GGS, et on construit un groupe GGS, qu'on note G_∞, qui est relié à G comme le décrit la proposition suivante :

Proposition 12.3. *Soit G un groupe proto-GGS, avec groupe de sommet A. Alors il existe un groupe G_∞ avec les propriétés suivantes :*

(1) *G_∞ est un groupe GGS agissant sur l'arbre Σ_∞^*. On suppose toujours $d \in \Sigma_\infty$, même si la cardinalité de ce dernier peut être plus petite que d. On note A_∞ le groupe de racine de G_∞ et ζ sa suite de récurrence ;*

(2) *A_∞ agit transitivement sur Σ_∞, et est engendré par les ζ_i, pour $i \in \Sigma_\infty \setminus \{d\}$;*

(3) *G_∞ est un groupe fractal ;*

(4) *pour tout $n \in \mathbb{N}$ suffisamment grand et tout sommet $\sigma \in \Sigma^n$, le groupe de sommet G^σ (défini à la page 20) est isomorphe soit à un sous-groupe de A, soit à un sous-groupe de $A \times \langle t\rangle \times G_\infty$ se surjectant sur le troisième facteur ;*

(5) *G_∞ est isomorphe à sous-groupe d'un quotient de G.*

Montrons tout d'abord l'utilité de ce résultat :

Corollary 12.4. *Soit G un groupe proto-GGS, et G_∞ le groupe qui lui est associé par la proposition 12.3. On a alors :*

(1) *G est infini si et seulement si G_∞ l'est ;*

(2) *G est de torsion si et seulement si G_∞ l'est ;*

(3) *G est de croissance intermédiaire si et seulement si G_∞ l'est ;*

(4) *G est à présentation finie si et seulement si G_∞ l'est ;*

On verra au chapitre suivant que ces propriétés sont bien plus faciles à vérifier sur G_∞ que sur G.

Démonstration. Montrons par exemple la première équivalence. Clairement si G est fini alors G_∞ l'est aussi. Réciproquement, on suppose G infini, et on considère $\mathsf{Stab}_G(n)$, pour n suffisamment grand, qui est infini et est un sous-groupe du produit direct des G^σ, pour $\sigma \in \Sigma^n$. L'un au moins de ces G^σ est donc infini, et donc le produit direct de $A \times \langle t\rangle$ avec G_∞ est infini. Il s'ensuit que G_∞ est infini. □

Démonstration de la proposition 12.3. Soit $\sigma \in \Sigma^n$ un sommet à distance n de la racine. Alors soit σ est dans la G-orbite du sommet d^n, auquel cas G^σ contient t ; soit il n'y est pas, auquel cas G^σ est un sous-groupe de A. En effet, si $\sigma = g \cdot d^n$, on a $t = \psi_\sigma(gtg^{-1}) \in G^\sigma$, et vice versa.

De même, G^σ est engendré par certains des ϵ_i, et éventuellement par t, donc est un sous-groupe de G. Notons A^σ le sous-groupe de G^σ engendré par les ϵ_i qui y résident.

Les A^σ sont naturellement arrangés en réseau, c'est-à-dire que pour tout σ préfixe de $\sigma\tau$ on a l'inclusion $A^{\sigma\tau} \leq A^\sigma$.

On a une tour décroissante de sous-groupes finis de A, qui doit donc se stabiliser à partir d'un certain rang, disons n. Pour tous les G^σ contenant t, donc dans l'orbite de d^n, les A^σ sont égaux, car les G^σ sont conjugués.

Posons $\Sigma_\infty = A^{d^n} \cdot d \subset \Sigma$. Alors A^{d^n} agit séparément sur Σ_∞ et sur $\Sigma \setminus \Sigma_\infty$; notons A_∞ le quotient de A^{d^n} agissant sur Σ_∞.

Pour tout $i \in \Sigma_\infty \setminus \{d\}$, on peut factoriser $\epsilon_i = \zeta_i\eta_i$, où ζ_i agit sur Σ_∞ et η_i agit sur son complémentaire. On définit alors G_∞ ainsi : c'est le groupe GGS agissant sur l'arbre $\mathcal{T}_{\Sigma_\infty}$, avec pour groupe de racine A_∞ et pour suite de récurrence $\zeta = (\zeta_i)$.

Par définition, A_∞ est transitif sur Σ_∞. S'il n'était pas engendré par les ζ_i, cela voudrait dire que le groupe $A^{d^{n+1}}$ serait un sous-groupe strict de A^{d^n}, ce qui contredirait le choix de n ; on l'a en effet supposé assez grand pour que la suite des A^{d^n} soit stable.

G_∞ est un groupe fractal : prenons par exemple le sur-groupe de sommet G_∞^d du sommet d. Il contient t_ζ, comme l'image de t_ζ par ψ_d, et aussi tous les ζ_i, comme images de t_ζ^a, où $a \in A_\infty$ est choisi pour envoyer d sur i. On a donc $G_\infty^d = G_\infty$, et de même pour tous les autres groupes de sommet de niveau 1. On applique alors le lemme 1.9.

Toujours pour n suffisamment grand et $\sigma \in \Sigma^n$, on a vu que si σ n'est pas dans la G-orbite de d^n, alors G^σ est un sous-groupe de A. Si σ appartient à l'orbite de d^n, il est en revanche isomorphe à G^{d^n}. Ce dernier groupe se décompose en sous-produit direct, selon sa partie agissant sur le sous-arbre $\Sigma_\infty\Sigma^*$ et celle agissant sur $(\Sigma \setminus \Sigma_\infty)\Sigma^*$. La première n'est autre que G_∞ ; la seconde est un sous-groupe de $A \times \langle t\rangle$ car ces générateurs commutent quand on les restreint à $(\Sigma \setminus \Sigma_\infty)\Sigma^*$.

Finalement, on peut considérer le sous-groupe de G_∞ engendré par t_ϵ et les ϵ_i pour $i \in \Sigma_\infty$, et le projeter sur G_∞ en envoyant t_ϵ sur t_ζ et ϵ_i sur ζ_i. □

La preuve donnée ci-dessus indique un algorithme, étant donné un groupe G défini par A et $(\epsilon_1, \dots, \epsilon_{d-1})$, qui construit un groupe GGS, G_∞, partageant de nombreuses propriétés avec G. Nous explicitons cet algorithme comme suit :

- Soit G un groupe GGS avec groupe de racine A et suite de récurrence ϵ. On construit la suite de sous-groupes de A suivante : $A_0 = A$ puis par induction
$$A_{n+1} = \big\langle \epsilon_i : i \in A_n \cdot d \setminus \{d\}\big\rangle.$$
- Soit A_∞ la limite $\bigcap A_n$ de cette tour de sous-groupes, et soit Σ_∞ l'orbite de d par A_∞.
- Pour tous les $i \in \Sigma_\infty \setminus \{d\}$, on factorise ϵ_i comme $\zeta_i\eta_i$, où ζ_i est une permutation de Σ_∞ et η_i est une permutation de $\Sigma \setminus \Sigma_\infty$.
- Le groupe GGS G_∞ agit sur l'arbre $|\Sigma_\infty|$-régulier $\mathcal{T}_{\Sigma_\infty}$, a pour groupe de racine A_∞, et pour suite de récurrence $\zeta_{i\in\Sigma_\infty\setminus\{d\}}$.

Pour illustrer cet algorithme, considérons G un groupe proto-GGS engendré par $A = \mathbb{Z}/d\mathbb{Z}$ et t_ϵ. Si les ϵ_i engendrent le sous-groupe n_1A de A, on a l'inclusion $\mathsf{Stab}_G(1) \subset G_1^d$, où $G_1 = \langle t_\epsilon, n_1A\rangle$. Ensuite, si $\{\epsilon_{n_1}, \epsilon_{2n_1}, \dots, \epsilon_{d-n_1}\}$ engendre le sous-groupe n_2A de A, il vient $\mathsf{Stab}_{G_1}(1) \subset (A^{n_1-1} \times G_2)^{d/n_1}$, où $G_2 = \langle t_\epsilon, n_2A\rangle$, et ainsi de suite.

CHAPITRE 13

Propriétés algébriques

Dans ce chapitre on décrit quelques propriétés algébriques des groupes GGS, et on donne des critères pour les propriétés suivantes : être de torsion, être infini, être de présentation finie, être de centre trivial. La propriété d'être de croissance intermédiaire sera étudiée au chapitre suivant.

Les deux premières propriétés — être de torsion et être infini — avaient été étudiées par Rostislav Grigorchuk dans le cas à d est premier et A cyclique, c'est-à-dire quand G est un sous-groupe du pro-d-Sylow $\mathsf{Aut}_*(\mathcal{T})$. Il a donné des critères faciles à vérifier dans [**Gri00**], qu'on redémontrera en corollaires. Ces critères ont été généralisés à d une puissance de premier (toujours pour A cyclique) par Taras Vovkivsky [**Vov98**], mais leur formulation est assez lourde.

Definition 13.1. Soit $H = \mathsf{Stab}_G(1)$ le stabilisateur du premier niveau. L'application ψ introduite en (6, page 19) donne, par restriction, une flèche

$$\psi : \begin{cases} H & \to G_1 \times \cdots \times G_d \\ h & \mapsto (\phi_1^{-1}(h), \dots, \phi_d^{-1}(h)) \\ ata^{-1} & \mapsto (\epsilon_{a(1)}, \dots, \epsilon_{a(d)}), \end{cases} \tag{18}$$

où par convention on a $\epsilon_d = t$. En d'autres termes, ψ identifie un élément stabilisant le premier niveau à un d-uple d'éléments des G_i.

Theorem 13.2. *Si G est un groupe GGS avec $|\Sigma| \geq 2$, alors G est fractal et infini.*

Corollary 13.3. *Soit G un groupe proto-GGS pour d premier et A cyclique. Alors G est infini si et seulement si $\epsilon_i \neq 1$ pour un i.*

Démonstration. La suite descendante (A_n) construite dans la proposition 12.3 satisfait soit $A_n = A$ soit $A_n = 1$, pour tous les $n \geq 1$. Dans le premier cas, $G_\infty = G$, alors que dans le second $G_\infty = 1$. □

Démonstration du théorème. Si $|\Sigma| \geq 2$, alors G a un sous-groupe d'indice $|A|$, donc au moins d, qui se surjecte sur G, par exemple par ψ_d. Il est donc infini. Une simple consultation de (18) montre que G est fractal. □

Pour le résultat suivant, on a besoin d'une notation supplémentaire : pour $\Pi \in \mathfrak{S}_\Sigma$, on note $s(\Pi) = \epsilon_{\Pi(d)} \cdots \epsilon_{\Pi^\ell(d)} \in A$, où $\ell + 1$ est la période de d sous l'action de Π (on a donc $\Pi^{\ell+1}(d) = d$). On remarque que $s(1) = 1$.

Theorem 13.4. *G est un groupe de torsion si et seulement si pour tout $\Pi \in A$ on a $s^n(\Pi) = 1$ pour un $n \in \mathbb{N}$ assez grand.*

Corollary 13.5. *Soit G un groupe GGS pour d premier et A cyclique. Alors G est de torsion si et seulement si $\epsilon_1 \dots \epsilon_{d-1} = 1$.*

Démonstration. Comme A est simple et abélien, il suffit de tester la condition $s^n(\Pi) = 1$ sur un générateur, disons $(1, 2, \dots, d)$, de A. Elle équivaut alors à la condition du théorème. □

On aura aussi besoin d'une notion de longueur pour G. Pour ce faire, on prend comme système générateur $A \cup \langle t \rangle$, et on note $|g|$ la longueur de l'élément g pour ce choix de générateurs.

LEMMA 13.6. *Soit* $g = (g_1, \ldots, g_d) \in \mathsf{Stab}_G(1)$. *Alors* $|g_i| \leq \frac{1}{2}(|g| + 1)$ *pour tout* i.

DÉMONSTRATION. On choisit une forme minimale pour g, c'est-à-dire un mot W tel que $|W| = |g|$. Au plus $\frac{1}{2}(|g| + 1)$ des lettres de W sont des t, et elles seules peuvent contribuer une lettre à g_i. □

DÉMONSTRATION DU THÉORÈME. Pour la nécessité de la condition : soit $\Pi \in A$ tel que $s^n(\Pi) \neq 1$ pour tout $n \in \mathbb{N}$. On considère l'élément $g = \Pi t$. Si ℓ est l'ordre de Π, on a $g^\ell = (*, \ldots, *, s(\Pi)T)$ dans $\mathsf{Stab}_G(1)$, où T est un produit de conjugués de t, et g^ℓ est non-trivial car $s(\Pi) \neq 1$. Continuant, si ℓ' est l'ordre de $s(\Pi)$, on a $g^{\ell\ell'} = (*, \ldots, *, (*, \ldots, *, s^2(\Pi)T'))$ dans $\mathsf{Stab}_G(2)$, où T' est de nouveau un produit de conjugués de t, avec $g^{\ell\ell'}$ non-trivial car $s^2(\Pi) \neq 1$. On continue de même.

Pour la suffisance de la condition : soit $g = (g_1, \ldots, g_d)\Pi$ dans G, avec $\Pi \in A$. Si $\Pi = 1$, on sait par induction que les g_i sont d'ordre fini, car ils sont plus courts que g par le lemme 13.6. Sinon, soit ℓ l'ordre de $\Pi_{|\Pi(d)}$, et considérons g^ℓ. Même s'il n'appartient pas à $\mathsf{Stab}_G(1)$, on peut considérer, pour tout $i \in \Pi(d)$, sa restriction g' au sous-arbre $i\Sigma^*$, et on a $g' = (g'_1, \ldots, g'_d)s(\Pi)$. De surcroît, la longueur de g' ne peut pas dépasser celle de g, car tous les symboles t de g' doivent provenir de symboles t différents dans g. On continue ainsi jusqu'à ce qu'on ait $s^n(\Pi) = 1$, auquel cas on retombe dans le cas traité au début, où $g^{(n)} \in \mathsf{Stab}_G(1)$ et donc $g_i^{(n)}$ est plus court que g par le lemme 13.6, donc de torsion par induction. □

THEOREM 13.7. *G est de présentation finie si et seulement si* $|\Sigma| \leq 2$.

DÉMONSTRATION. Si $|\Sigma| = 2$, alors G est diédral infini, donc de présentation finie. On suppose donc $|\Sigma| \geq 3$, et on montre que G n'est pas de présentation finie.

Soit $\langle A, t | \mathcal{R}\rangle$ une présentation de G pour un ensemble fini $\mathcal{R}$ de relateurs. On suppose de plus que la longueur totale des relateurs est minimale.

L'application ψ décrite en (18) peut être considérée comme une substitution sur l'alphabet $S = A \cup \langle t\rangle$ engendrant G ; on note T l'ensemble des mots sur S s'évaluant à un élément de H dans G, et $\tilde{\psi}$ l'application $T \to (S^*)^d$ induite par ψ.

Chacun des relateurs $r \in \mathcal{R}$, s'évaluant en 1 dans G, est à fortiori un mot dans T. On peut donc considérer $\tilde{\psi}(r) = (r_1, \ldots . r_d)$. Par le lemme 13.6, toutes les relations r_i sont plus courtes que r, si $|r| \geq 2$.

Maintenant, $\langle S | \, r_d$ pour tout $r \in \mathcal{R}\rangle$ est une présentation du groupe de sommet G^d, isomorphe à G, dont la longueur totale des relations est plus petite. On obtient une contradiction avec la minimalité supposée de $\mathcal{R}$, à moins que tous les relateurs dans $\mathcal{R}$ soient de longueur 1.

Comme G est supposé fractal, t est d'ordre d. S'il existe un ϵ_i dans la suite de récurrence de G d'ordre j divisant strictement d, on choisit un élément a de A envoyant d sur j, et on considère $x = [t^j, t^a]$. Ce mot appartient à T, et pour tout $i \neq d$ on a $\tilde{\psi}_i(x) \in A$. Comme A est d'ordre fini, on a même $\tilde{\psi}_i(x^n) = 1$ pour tout i, si on prend $n = |A|$. On a donc $[x^n, (x^n)^a] = 1$ dans G, ce qui est une relation non-triviale et de longueur strictement plus grande que 1.

Finalement, on suppose que tous les ϵ_i sont d'ordre d. On utilise seulement ici l'hypothèse que $|\Sigma| \geq 3$. Soit $a \in A$ envoyant d sur 1 ; alors $x = [t, t^a]$ satisfait $\tilde{\psi}_i(x^n) = 1$ pour tout $i \in \Sigma \setminus \{d, 1\}$. Soit $b \in A$ envoyant d sur 2, et $y = [t, t^b]$. Si on pose $z = [x^n, y^n]$, on a $\tilde{\psi}_i(z) = 1$ pour tout $i \in \Sigma \setminus \{d\}$. Finalement, $[z, z^a] = 1$ est une relation non-triviale et de longueur strictement plus grande que 1. □

On remarque en passant qu'il existe deux stratégies pour montrer que de tels groupes ne sont pas de présentation finie. L'une, appliquée avec succès au groupe de Grigorchuk $\mathfrak{G}$, consiste à construire des relateurs de plus en plus complexes eu moyen de substitutions analogues à la fonction σ de la section 6.2. La seconde, utilisée ici, a été introduite par Rostislav Grigorchuk dans **[Gri80a]**

Rostislav Grigorchuk a conjecturé par ailleurs qu'aucun groupe de croissance intermédiaire ne peut admettre de présentation finie. Cette question est toujours ouverte.

THEOREM 13.8. *$Z(G)$ est trivial dès que $|\Sigma| \geq 2$.*

DÉMONSTRATION. On suppose que G est non-trivial. Soit $g \in Z(G)$. On montre d'abord que $g \in \mathsf{Stab}_G(1)$. En effet, écrivons $g = ha$ avec $h \in \mathsf{Stab}_G(1)$ et $a \in A$. Si $a(d) = d$ et $a(i) = j$ pour un $i \neq j$, alors g ne commute pas avec les éléments a' de A tels que $a'(q) = i$. Si $a(q) = i \neq q$ alors g ne commute pas avec t, car $\psi_q([g,t]) = a't^{-1}$ pour un $a' \in A$.

On procède maintenant par induction sur $|g|$. Si $|g| = 1$, on a $g \in A \cup \langle t \rangle$ et la discussion ci-dessus montre que g n'est pas central. On suppose donc $|g| \geq 2$. Si $g \in \mathsf{Stab}_G(1)$, on considère $\psi(g) = (g_1, \ldots, g_d)$ où chacun des g_i est plus court que g par le lemme 13.6, et l'induction peut être appliquée. Si $g \notin \mathsf{Stab}_G(1)$, le paragraphe précédent montre que g ne peut pas être central. □

On a aussi $\mathsf{Cent}_{\mathsf{Aut}\,\mathcal{T}}(G) = 1$ (ce qui est une affirmation plus faible que le théorème ci-dessus) dans un cadre plus général des groupes spéciaux de la définition 2.7 — voir le théorème 6 dans [**Gri00**].

PROPOSITION 13.9. *G est résiduellement fini.*

DÉMONSTRATION. Comme G est un sous-groupe de $\mathsf{Aut}\,\mathcal{T}$, il suffit de le montrer pour $\mathsf{Aut}\,\mathcal{T}$. Mais on a vu au chapitre 1 que $\mathsf{Aut}\,\mathcal{T}$ est approximé par ses quotients finis $\mathsf{Aut}\,\mathcal{T}/\,\mathsf{Stab}(n)$. □

THEOREM 13.10. *Si G est infini, alors il contient des éléments d'ordre orbitrairement grand (éventuellement infini).*

DÉMONSTRATION. Soit $a \in A$ agissant non-trivialement sur d, et soit n la longueur de l'orbite de d par a. Considérons le sous-groupe $G^{(n)}$ engendré par les puissances n-ièmes d'éléments de G, et $I = G^{(n)} \cap \mathsf{Stab}_G(d)$.

Ce sous-groupe I contient $(at_\epsilon)^n$, et est normal dans $\mathsf{Stab}_G(d)$; ainsi son image par ϕ_d est un sous-groupe normal J de G contenant bt_ϵ pour un certain $b \in A$. De plus, G/J est un quotient de A, donc J est d'indice au plus $|A|$ dans G et contient $\mathsf{Stab}_G(1)$, puisque son action sur Σ^* peut se lire sur Σ. Ainsi I contient $\mathsf{Stab}_G(2)$.

Supposons maintenant par l'absurde que G est d'exposant N ; ainsi $\mathsf{Stab}_G(2)$ est d'exposant au moins N, puisqu'il se surjecte sur G, et I est d'exposant au moins N. puisqu'il contient $\mathsf{Stab}_G(2)$. Il suit que G est d'exposant au moins nN, ce qui est une contradiction. □

On introduit un affaiblissement de la notion de groupe branché :

DEFINITION 13.11. Soit G un groupe agissant sur l'arbre $\mathcal{T}$. Il est *presque à branches* sur son sous-groupe K s'il agit sphériquement transitivement, et si $K \times \cdots \times K$ est un sous-groupe de $\psi(K)$.

On remarque que tout groupe sphériquement transitif est presque à branches sur 1, son sous-groupe trivial. S'il est presque à branches sur K et que celui-ci est d'indice fini dans G, alors G est à branches. Clairement, si K est non-trivial alors il est infini.

THEOREM 13.12. *Soit G un groupe GGS avec $|\Sigma| \geq 3$. Alors il est presque à branches sur*

$$K = \big((G')^{(|A|)}\big)',$$

où $H^{(n)}$ désigne le sous-groupe de H engendré par les puissances n-ièmes de ses éléments, et K n'est pas trivial.

DÉMONSTRATION. G' contient tous les éléments de la forme $[t, t^a]$, qui peuvent s'écrire $(a_1, \ldots, [a_i, t], a_{i+1}, \ldots, [t, a_d])$. Notons $n = |A|$. Ainsi, $(G')^{(|A|)}$ contient tous éléments s'écrivant $(1, \ldots, [a_i, t]^n, 1, \ldots, [t, a_d]^n)$, et finalement son groupe dérivé contient tous les éléments de la forme $(1, \ldots, [[t, a]^n, [t, a']^n])$. On a ainsi $1 \times \cdots \times 1 \times K \subset K$.

Pour montrer que K n'est pas trivial, on argumente d'abord que G' est non-trivial, car d'indice fini dans un groupe infini ; ensuite, par le théorème 13.10, que $(G')^{(n)}$ est non-trivial ; ce dernier groupe, par la preuve du théorème 13.10, admet même un sous-groupe se surjectant sur G, et ne peut donc en aucun cas être abélien. $\square$

CHAPITRE 14

Croissance des mots

Dans une large mesure, l'intérêt pour les groupes GGS a été amorcé par la recherche de groupes de croissance intermédiaire. On montre dans ce chapitre qu'essentiellement tous les groupes GGS sont de croissance intermédiaire.

THÉORÈM 14.1. *Soit G un groupe GGS. Alors G est de croissance superpolynômiale si et seulement si $|\Sigma| \geq 3$.*

DÉMONSTRATION. Si $|\Sigma| \leq 2$, alors G est soit le groupe trivial, soit le groupe diédral infini ; dans tous les cas, il est de croissance polynômiale.

Supposons à présent $|\Sigma| \geq 3$. Par le théorème 13.12, G est presque à branches sur un sous-groupe K infini.

Supposons par l'absurde que K est de croissance polynômiale, disons $\gamma_K \sim n^D$. Comme $K^d < \psi(K)$, on a $n^{dD} \sim \gamma_{K^d} \precsim \gamma_K \sim n^D$, ce qui est une contradiction. Il suit que K est de croissance superpolynômiale.

Comme G contient K, on a $\gamma_G \succsim \gamma_K$ et donc G est aussi de croissance superpolynômiale. $\square$

THEOREM 14.2. *Soit G un groupe GGS. Alors G est de croissance sous-exponentielle.*

On fixe une fois pour toutes le système de générateurs de G : il s'agit de

$$S = A\langle t\rangle = \{at^i |\, a \in A, 0 \leq i < d\}.$$

On note $|g|$ la longueur minimale de l'élément $g \in G$ en tant que mot sur l'alphabet S, et aussi $|w|$ la longueur du mot $w \in S^*$. À au plus 1 près (correspondant à un élément de A placé à la fin), $|g|$ est le nombre de sous-mots maximaux de w constitués uniquement de t et de t^{-1}, minimal dans une écriture de g.

On a une flèche naturelle $\pi : S^* \twoheadrightarrow G$, qui correspond à l'évaluation dans G. Elle admet une rétraction $\nu : G \to S^*$, qui est en fait un choix de forme normale minimale pour G. Bien évidemment ν n'est pas unique.

DEFINITION 14.3. Une *fusion élémentaire* sur un mot w est une application d'une des transformations suivantes, pour $a, b \in A$:

$$at^0 \cdot bt^i \rightsquigarrow abt^i$$
$$at^i \cdot 1t^j \rightsquigarrow at^{i+j}$$

diminuant de 1 la longueur de w. On note $\rho : S^* \to S^*$ la clôture itérative de cette opération.

Par exemple, supposons $d = 5$ et $A = \mathbb{Z}/5\mathbb{Z} = \langle a\rangle$. Alors, en tant que mots, on a $|a| = 1$, $|at| = 1$ et $|at^2a^3t^4a^5ta^2t| = 4$. Une fusion peut se produire dans ce dernier mot, donnant le mot $at^2a^3t^4ta^2t$ de norme 3 ; une nouvelle fusion donne le mot $at^2a^3a^2t$ de norme 2, et une dernière fusion donne at^3 de norme 1. Il s'agit là de la longueur minimale de l'élément de départ, et on a donc $\rho(at^2a^3t^4ta^2t) = at^3$.

Dans la plupart des cas, un mot peut être raccourci en utilisant d'autres opérations que la fusion. En fait, la réduction par fusion n'est autre que la mise sous forme normale dans le groupe $A * \langle t\rangle$, dont G est un quotient — on prouve ainsi que l'opération ρ est bien définie, c'est-à-dire que l'ordre dans lequel on exécute les fusions sur un mot n'a pas d'influence sur le résultat final.

L'application ψ définie en (18, page 73) se relève en une application

$$\tilde{\psi}: \begin{cases} \pi^{-1}(H) & \to (S^*)^d \\ w & \mapsto (w_1, \dots, w_d) \end{cases}$$

sur S^*, satisfaisant le diagramme

$$\begin{tikzcd} \pi^{-1}(H) \arrow[r, "\tilde{\psi}"] \arrow[d, "\pi"] & (S^*)^d \arrow[d, "\pi"] \\ H \arrow[r, "\psi"'] & G^d \end{tikzcd}$$

Lemma 14.4. *On a, pour tout $\eta \in (0,1)$,*

$$\lim_{n\to\infty} \binom{n}{(1-\eta)n}^{1/n} = \frac{1}{\eta^\eta(1-\eta)^{1-\eta}} =: C(\eta).$$

Démonstration. On écrit

$$\binom{n}{(1-\eta)n} = \frac{n!}{(\eta n)!((1-\eta)n)!}.$$

Par la formule de Stirling [**GKP94**, équation (9.29)], on a les approximations $\sqrt[n]{n!} \approx n/e$, $\sqrt[n]{(\eta n)!} \approx (\eta n/e)^\eta$, et $\sqrt[n]{((1-\eta)n)!} \approx ((1-\eta)n/e)^{1-\eta}$, d'autant meilleures quand $n \to \infty$. $\square$

Lemma 14.5. *Pour tout $a, b \in \mathbb{R}_+$, on a*

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} = \max\{a, b\}.$$

Démonstration. Sans perte de généralité, on suppose $a \geq b$; on a alors, pour $x = b/a \leq 1$,

$$a \leq (a^n + b^n)^{1/n} = a\sqrt[n]{1 + x^n} \leq a(1 + x/n) \to a.$$

$\square$

Démonstration du théorème 14.2. On va montrer qu'il y a un nombre sous-exponentiel d'éléments de longueur donnée ; plus précisément, que

$$\lambda = \limsup_{n\to\infty} \sqrt[n]{|\{g \in G|\,|g| \leq n\}|} = 1.$$

Soit K un entier et $\eta \in (0,1)$ à fixer plus tard. Soit $V \subset \mathsf{Stab}_G(K)$ l'ensemble des éléments de longueur au plus n et fixant le K-ième niveau, et soit $W = \nu^{-1}(V)$ un ensemble de mots de longueur minimale leur correspondant. On veut estimer $|W|$; pour ce faire, soit $w \in W$, et notons $\tilde{\psi}^K(w) = (w_{1\dots 1}, \dots, w_{d\dots d})$. Alors

$$\sum_{\sigma\in\Sigma^K} |\rho(w_\sigma)| = |w| - \sum_{\sigma\in\Sigma^K} \text{nombre de fusions dans } w_\sigma.$$

On suppose par l'absurde que le taux de croissance λ de G est strictement plus grand que 1. Ainsi, pour tout $\epsilon > 0$, il existe un entier N tel que

$$(\lambda - \epsilon)^n < \gamma_G(n) < (\lambda + \epsilon)^n$$

est vérifié dès que $n \geq N$. On définit la partition suivante de W :

$$X = \Big\{w \in W \Big| \sum_{\sigma\in\Sigma^K} |\rho(w_\sigma)| < \eta|w|\Big\},$$

$$Y = \Big\{w \in W \Big| \sum_{\sigma\in\Sigma^K} |\rho(w_\sigma)| \geq \eta|w|\Big\}.$$

Ainsi X est composé des mots w ayant beaucoup de fusions dans leur décomposition de niveau K, et Y des mots ayant peu de fusions. On estime indépendamment les cardinalités respectives $\gamma_X(n)$ et $\gamma_Y(n)$ de X et Y.

Il y a un polynôme $P(n)$ tel que

$$\gamma_X(n) \leq P(n)(\lambda+\epsilon)^{\eta n} : \tag{19}$$

en effet, tout $w \in W$ est déterminé par les $\rho(w_\sigma)$, dont on note n_σ les longueurs normes respectives, et donc

$$\gamma_X(n) \leq \sum_{\{n_\sigma\}} \prod_{\sigma\in\Sigma^K} \gamma_G(n_\sigma),$$

où la somme parcourt les partitions entières de ηn en d^K parties. Quand $n_i < N$, les $\gamma_G(n_\sigma)$ ne peuvent contribuer qu'un facteur borné, $\gamma_G(N)$, au produit, et on a donc

$$\gamma_G(n_1)\cdots\gamma_G(n_d) < \gamma_G(N)^{d^K}(\lambda+\epsilon)^{\eta n}.$$

Le nombre de partitions de n en p parties est borné par le polynôme

$$P'_p(n) = \frac{n^{p-1}}{p!(p-1)!}$$

(voir, par exemple, [**LW92**, théorème 15.1]) et l'affirmation (19) est vraie pour le choix $P(n) = \gamma(N)^{d^K} P'_{p^K}(n)$.

Par ailleurs, on peut considérer au lieu de Y l'ensemble plus grand de tous les mots de S^* ayant au plus $(1-\eta)n$ fusions dans leur décomposition de niveau K. Si on définit Y_0 comme l'ensemble de tous les mots réduits de longueur au plus n (c'est-à-dire $Y_0 = \rho(S^n) \cap \pi^{-1}(\mathsf{Stab}_G(n))$), et inductivement

$$Y_{k+1} = \{w \in Y_k \,|\, \rho(\tilde\psi_i(w)) \in Y_k \text{ pour tout } i \in \Sigma\},$$

où le nombre total de fusions dans les $\tilde\psi_i(w)$ n'excède pas $(1-\eta)n$, on a $|Y_0| \leq |S|^n$ et

$$|Y_{k+1}| \leq \binom{n}{(1-\eta)n}\left(\frac{|A|-1}{|A|}\right)^{\eta n}\left(\frac{1}{|A|}\right)^{(1-\eta)n}|Y_k| \approx \left(C(\eta)\frac{|A|-1}{|A|}\right)^n |Y_k|$$

pour n suffisament grand et $\eta < 1$, par le lemme 14.4. En effet il y a $\approx n$ symboles 't' dans les $\tilde\psi_i(w)$ pour $w \in Y_k$, séparés par des symboles de A ; et il y a au plus $(1-\eta)n$ fusions quand au plus $(1-\eta)n$ de ces symboles de A sont triviaux, ce qui se produit le nombre de fois indiqué. Clairement, on a $|Y| \leq |Y_K|$, et ainsi

$$\gamma_Y(n) \leq \left(|S|C(\eta)^K\frac{(|A|-1)^K}{|A|^K}\right)^n. \tag{20}$$

Toujours pour des $n \geq N$, on combine (19) et (20) en écrivant

$$\begin{aligned}(\lambda-\epsilon)^n &\leq \gamma_G(n) \leq [G : \mathsf{Stab}_G(K)](\gamma_X(n)+\gamma_Y(n))\\ &\leq [G : \mathsf{Stab}_G(K)]\left(P(n)(\lambda+\epsilon)^{\eta n} + \left(|S|C(\eta)^K\frac{(|A|-1)^K}{|A|^K}\right)^n\right),\end{aligned}$$

on en prend la racine n-ième et on laisse n tendre vers l'infini. Utilisant le lemme 14.5, on obtient

$$\lambda - \epsilon \leq \max\left\{(\lambda+\epsilon)^\eta, |S|C(\eta)^K\left(\frac{|A|-1}{|A|}\right)^K\right\}.$$

Maintenant, on fait tendre ϵ vers 0 ; on prend $\eta < 1$ suffisament grand pour que $L := C(\eta)\frac{|A|-1}{|A|}$ soit strictement plus petit que 1 ; et on prend K assez grand pour qu'on ait $|S|L^K < \lambda$. On a ainsi obtenu

$$\lambda \leq \max\{\lambda^\eta, |S|L^K\},$$

ce qui est une contradiction. □

Annexes

Liste des publications

Le premier article auquel j'ai participé est ń Estimates for simple random walks on fundamental groups of surfaces ż, [**BCCH97**], dans lequel on estime par divers moyens le rayon spectral de la marche aléatoire simple sur le groupe fondamental J_g d'une surface orientable de genre g donné par la présentation

$$J_g = \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, [a_1, b_1] \dots [a_g, b_g] = 1 \right\rangle.$$

Le résultat principal obtenu dans cet article est l'encadrement du rayon spectral (défini en 9.1) de J_g, pour le système de générateurs fixé ci-dessus :

$$\frac{\sqrt{4g-1}}{2g} < \nu(J_g) \leq \frac{\sqrt{4g-2}+\frac{1}{2}}{2g},$$

où la première inégalité est due à Harry Kesten [**Kes59**].

Toujours avec pour but d'obtenir des approximations de $\nu(J_g)$, j'ai écrit un court texte améliorant la borne inférieure de $\nu(J_g)$ [**Bar96**], qui a fait l'objet d'une présentation à la conférence de Castelvecchio en Italie. Le principe y était de compter un ensemble de courbes fermées dans le graphe de Cayley de J_g ressemblant à des ń hydres ż, c'est-à-dire des chemins réductibles au chemin trivial par suppression d'aller-retours, à ceci près qu'ils ont des ń têtes ż entourant des cellules élémentaires du graphe, qu'il faut ń couper ż pour pouvoir réduire le chemin. À titre d'exemple, la borne inférieure dûe à Harry Kesten donne $\nu(J_2) > 0.66143$; la croissance des ń hydres ż donne

$$\nu(J_2) \geq 0.6623.$$

Plus tard, j'ai introduit une autre classe de chemins fermés dans le graphe de Cayley de J_2, que j'ai appelé des ń feuilles de houx ż [**Bar96**] et qui englobe la classe des ń hydres ż. Essentiellement, les ń feuilles de houx ż sont des structures récursives obtenues par imbrication d'hydres. J'obtiens ainsi

$$\nu(J_2) \geq 0.6624.$$

L'amélioration peut sembler insignifiante, mais il semble que cette approximation est très bonne — il y a beaucoup plus à gagner du côté de la borne supérieure.

Dès mes débuts d'assistanat (1995), j'ai collaboré avec Tullio Ceccherini-Silberstein, en particulier sur les différents problèmes de comptage qu'on rencontre dans le graphe de Cayley de J_g. Nous avons écrit un bref article donnant des résultats sur la croissance de ces graphes, qui est reproduit à l'annexe C.

Le premier article mathématique que j'ai publié seul est [**Bar99**]. Il est l'objet de la deuxième partie de cette thèse.

J'ai ensuite obtenu des approximations de la croissance du groupe de Grigorchuk $\mathfrak{G}$, tant par le haut [**Bar98**] que par le bas [**Bar00d**]. Ces deux articles sont résumés à la section 6.5, et sont reproduits aux annexes A et B.

Après la publication de [**Bar98**], je suis entré en contact avec Zoran Šunić avec lequel j'ai étendu les résultats de [**Bar98**] à d'autres groupes. L'article écrit en commun [**BŠ00**] est résumé au chapitre 7.

J'ai eu la grande chance de rencontrer Rostislav Grigorchuk en 1993, et de profiter depuis de ses nombreux conseils et idées. Nous avons ainsi écrit l'article [**BG00a**] sur les algèbres de Lie associées à son groupe, qui est résumé à la section 6.4 et est reproduit à l'annexe D. Nous avons ensuite écrit deux articles [**BG99b, BG99a**] étudiant le spectre de représentations quasi-régulières associées à des groupes fractals, qui sont résumés aux chapitres 5 et 4, et reproduits aux annexes E et F.

J'ai aussi découvert il y a cinq ans (en 1995) une propriété curieuse de polynômes sur le corps à deux éléments, et ce, en connexion avec un problème de combinatoire paru à l'Olympiade de Mathématiques de 1993. Je n'ai rédigé ces recherches qu'en 1999 [**Bar00b**].

Enfin, dans un domaine un peu plus éloigné, j'ai obtenu à l'aide d'un ordinateur deux grilles de mots croisés de taille 9×9 sans aucune case noire [**BB96**]. Je m'autorise à citer cet article parce qu'il est paru dans une revue mathématique.

ANNEXE A

The Growth of Grigorchuk's Torsion Group

Laurent Bartholdi

A.1. Introduction

The notion of growth for finitely generated groups was introduced in the 1950's in the former Ussr [**Sva55**] and in the 1960's in the West [**Mil68a**]. There are well-known classes of groups of polynomial growth (abelian, and more generally virtually nilpotent groups [**Gro81a**]) and of exponential growth (non-virtually-nilpotent linear [**Tit72**] or non-elementary hyperbolic [**GH90**] groups). However, the first example of a group of intermediate growth was discovered later, by Rostislav Grigorchuk; see [**Gri83, Gri84, Gri91**]. He showed that the growth γ of his group satisfies

$$e^{\sqrt{n}} \precsim \gamma(n) \precsim e^{n^\beta},$$

where $\beta = \log_{32}(31) \approx 0.991$; see below for the precise definition of growth. The purpose of this note is to prove the following improvement:

Theorem A.1. *Let η be the real root of the polynomial $X^3 + X^2 + X - 2$, and set $\alpha = \log(2)/\log(2/\eta) \approx 0.767$. Then the growth γ of Grigorchuk's group satisfies*

$$e^{\sqrt{n}} \precsim \gamma(n) \precsim e^{n^\alpha}.$$

A.2. Growth of Groups

Let G be a group generated as a monoid by a finite set S. A *weight* on (G, S) is a function $\omega : S \to \mathbb{R}_+^*$. It induces a *length* ∂_ω on G by

$$\partial_\omega : \begin{cases} G \to \mathbb{R}_+ \\ g \mapsto \min\{\omega(s_1) + \cdots + \omega(s_n) |\, s_1 \cdots s_n =_G g\}. \end{cases}$$

A *minimal form* of $g \in G$ is a representation of g as a word of minimal length over S. The *growth* of G with respect to ω is then

$$\gamma_\omega : \begin{cases} \mathbb{R}_+ \to \mathbb{R}_+ \\ n \mapsto \#\{g \in G |\, \partial_\omega(g) \le n\}. \end{cases}$$

The function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is *dominated* by $\delta : \mathbb{R}_+ \to \mathbb{R}_+$, written $\gamma \precsim \delta$, if there is a constant $C \in \mathbb{R}_+$ such that $\gamma(n) \le \delta(Cn)$ for all $n \in \mathbb{R}_+$. Two functions $\gamma, \delta : \mathbb{R}_+ \to \mathbb{R}_+$ are *equivalent*, written $\gamma \sim \delta$, if $\gamma \precsim \delta$ and $\delta \precsim \gamma$.

The following lemmata are well known:

Lemma A.2. *Let S and S' be two finite generating sets for the group G, and let ω and ω' be weights on (G, S) and (G, S') respectively. Then $\gamma_\omega \sim \gamma_{\omega'}$.*

Proof. Let $C = \max_{s \in S} \partial_{\omega'}(s)/\omega(s)$. Then $\partial_{\omega'}(g) \le C\partial_\omega(g)$ for all $g \in G$, and thus $\gamma_\omega(n) \le \gamma_{\omega'}(Cn)$, from which $\gamma_\omega \precsim \gamma_{\omega'}$. The opposite relation holds by symmetry. □

The *growth type* of a finitely generated group G is the $\sim$-equivalence class containing its growth functions; it will be denoted by γ_G.

Note that all exponential functions b^n are equivalent, and polynomial functions of different degree are inequivalent; the same holds for the subexponential functions e^{n^α}. We have

$$0 \precsim\!\!\!\!\!\not\,\,\, n \precsim\!\!\!\!\!\not\,\,\, n^2 \cdots \precsim\!\!\!\!\!\not\,\,\, e^{\sqrt{n}} \precsim\!\!\!\!\!\not\,\,\, \cdots \precsim\!\!\!\!\!\not\,\,\, e^n.$$

Note also that the ordering $\precsim$ is not linear.

LEMMA A.3. *Let G be a finitely generated group. Then $\gamma_G \precsim e^n$.*

PROOF. Choose for G a finite generating set S, and define the weight ω by $\omega(s) = 1$ for all $s \in S$. Then $\gamma_\omega(n) \leq |S|^n$ for all n, so $\gamma_G \precsim e^n$. $\square$

If there is a $d \in \mathbb{N}$ such that $\gamma_G \precsim n^d$, the group G is of *polynomial growth* of degree at most d; if $\gamma_G \sim e^n$, then G is of *exponential growth*; otherwise G is of *intermediate growth*. The existence of groups of intermediate growth was first shown by Grigorchuk [**Gri83**].

A.3. The Grigorchuk 2-group

Let Σ^* be the set of finite sequences over $\Sigma = \{0,1\}$. For $x \in \Sigma$ set $\overline{x} = 1-x$. Define recursively the following length-preserving permutations of Σ^*:

$$a(x\sigma) = \overline{x}\sigma;$$
$$b(0\sigma) = 0a(\sigma), \qquad b(1\sigma) = 1c(\sigma);$$
$$c(0\sigma) = 0a(\sigma), \qquad c(1\sigma) = 1d(\sigma);$$
$$d(0\sigma) = 0\sigma, \qquad d(1\sigma) = 1b(\sigma).$$

Then G, the Grigorchuk 2-group [**Gri80a, Gri84**], is the group generated by $S = \{a,b,c,d\}$. It is readily checked that these generators are of order 2 and that $V = \{1,b,c,d\}$ is a Klein group.

Let $H = V^G$ be the normal closure of V in G. It is of index 2 in G and preserves the first letter of sequences; i.e. $H \cdot x\Sigma^* \subset x\Sigma^*$ for all $x \in \Sigma$. There is a map $\psi : H \to G \times G$, written $g \mapsto (g_0, g_1)$, defined by $0g_0(\sigma) = g(0\sigma)$ and $1g_1(\sigma) = g(1\sigma)$. As $H = \langle b,c,d,b^a,c^a,d^a\rangle$, we can write ψ explicitly as

$$\psi : \begin{cases} b \mapsto (a,c), & b^a \mapsto (c,a) \\ c \mapsto (a,d), & c^a \mapsto (d,a) \\ d \mapsto (1,b), & d^a \mapsto (b,1). \end{cases}$$

A.4. The Growth of G

Let $\eta \approx 0.811$ be the real root of the polynomial $X^3 + X^2 + X - 2$, and define the following function on S:

$$\omega(a) = 1 - \eta^3 = \eta^2 + \eta - 1, \qquad \omega(c) = 1 - \eta^2,$$
$$\omega(b) = \eta^3 = 2 - \eta - \eta^2, \qquad \omega(d) = 1 - \eta.$$

It is a weight, because it takes positive values on every generator.

LEMMA A.4. *Every $g \in G$ admits a minimal form*

$$[*]a * a * a \cdots * a[*],$$

where $* \in \{b,c,d\}$ and the first and last $*$s are optional.

PROOF. Clearly $\omega(s) > 0$ for $s \in S$, so ω is a weight. Let w be a minimal form of g. The lemma asserts that one can suppose there are no consecutive letters in $\{b,c,d\}$ in w; now two equal consecutive letters cancel, and the product of any two distinct letters in $\{b,c,d\}$ equals the third one. For any arrangement (x,y,z) of $\{b,c,d\}$ we have $\omega(x) \leq \omega(y) + \omega(z)$, so the substitution of z for xy will not increase the weight of w. $\square$

PROPOSITION A.5. *Let $g \in H$, with $\psi(g) = (g_0, g_1)$. Then*

$$\eta\big(\partial_\omega(g) + \omega(a)\big) \geq \partial_\omega(g_0) + \partial_\omega(g_1).$$

PROOF. Let w be a minimal form of g. Thanks to Lemma A.4 we may suppose the number of $*$s in w is at most the number of as plus one. Construct words w_0, w_1 over S using ψ seen as a substitution on words; they represent g_0 and g_1 respectively. Note that

$$\begin{aligned}\eta(\omega(a) + \omega(b)) &= \omega(a) + \omega(c),\\ \eta(\omega(a) + \omega(c)) &= \omega(a) + \omega(d),\\ \eta(\omega(a) + \omega(d)) &= 0 + \omega(b).\end{aligned}$$

As $\psi(b) = (a, c)$ and $\phi(aba) = (c, a)$, each b in w contributes $\omega(a) + \omega(c)$ to the total weight of w_0 and w_1; the same argument applies to c and d. Now, grouping together pairs of $*$s in $\{b, c, d\}$ and as, we see that $\eta(\partial_\omega(g))$ is a sum of left-hand terms, possibly $-\eta\omega(a)$; while $\partial_\omega(g_0) + \partial_\omega(g_1)$ is bounded by the total weight of the letters in w_0 and w_1, which is precisely the sum of the corresponding right-hand terms. □

Let $\alpha = \log(2)/\log(2/\eta) \approx 0.767$, and for $n \in \mathbb{N}$ set $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number; remember that it is the number of labelled binary rooted trees with $n + 1$ leaves [**Cat38, LW92**, page 119].

PROPOSITION A.6. *Let $\zeta = \frac{\omega(a)}{2/\eta - 1}$, let $K > \zeta$ be any constant, and for $n \in \mathbb{R}_+$ set*

$$L_n = \max\left\{1, \left\lceil 2\left(\frac{n-\zeta}{K-\zeta}\right)^\alpha\right\rceil - 1\right\}.$$

Then we have

$$\gamma_\omega(n) \leq C_{L_n - 1} 2^{L_n - 1} \gamma_\omega(K)^{L_n}. \tag{21}$$

PROOF. We construct an injection ι of G into the set of labelled binary rooted trees each of whose leaves is labelled by an element of G of weight bounded by K and each of whose interior vertices is labelled by an element of the subgroup $\langle a\rangle$ of G. For $g \in G$, $\iota(g)$ is called its *representation.* It is constructed as follows: if $g \in G$ satisfies $\partial_\omega(g) \leq K$, its representation is a tree with one vertex labelled by g. If $\partial_\omega(g) > K$, let $h \in \langle a\rangle$ be such that $gh \in H$, and write $\psi(gh) = (g_0, g_1)$. By Proposition A.5, $\partial_\omega(g_i) \leq \eta\partial_\omega(g)$, so we may construct inductively the representations of g_0 and g_1. The representation of g is a tree with h at its root vertex and $\iota(g_0)$ and $\iota(g_1)$ attached to its two branches.

We first claim that ι is injective: let $\mathcal{T}$ be a tree in the image of ι. If $\mathcal{T}$ has one node labelled by g, then $\iota^{-1}(\mathcal{T}) = \{g\}$. If $\mathcal{T}$ has more than one vertex, let $h \in \langle a\rangle$ be the label of the root vertex and $(\mathcal{T}_0, \mathcal{T}_1)$ be the two subtrees connected to the root vertex. By induction on the number of vertices of $\mathcal{T}$, we have $\mathcal{T}_i = \iota(g_i)$ for unique g_0 and g_1. Then as ψ is injective there is a unique $g \in G$ with $\psi(gh) = (g_0, g_1)$, and $\iota^{-1}(\mathcal{T}) = \{g\}$.

We next prove by induction on n that if $\partial_\omega(g) \leq n$ then its representation is a tree with at most L_n leaves. Indeed if $n \leq K$ then g's representation has one leaf and $L_n = 1$, while otherwise g's representation is made up of those of g_0 and g_1. Say $\partial_\omega(g_0) = \ell$ and $\partial_\omega(g_1) = m$; then by Proposition A.5 we have $\ell + m \leq \eta(n + \omega(a))$. By induction these representations have at most L_ℓ and L_m leaves. As $\alpha < 1$, we have $L_\ell + L_m \leq 2L_{(\ell+m)/2}$ for all ℓ, m; and by direct computation, $L_{\eta/2(n+\omega(a))} = \lfloor L_n/2\rfloor$, so the number of leaves of g's representation is

$$L_\ell + L_m \leq 2L_{(\ell+m)/2} \leq 2L_{\eta/2(n+\omega(a))} \leq L_n,$$

as was claimed.

We conclude that $\gamma(n)$ is bounded by the number of representations with L_n leaves; there are C_{L_n-1} binary trees with L_n leaves, 2 choices of labelling for each of the $L_n - 1$ interior vertices, and $\gamma(K)$ choices for each leaf; so Equation (21) follows. □

A lower bound on the growth of G comes from the fact that G is residually a 2-group:

THEOREM A.7 (Grigorchuk [**Gri89**]). *Suppose G is a finitely generated residually-p group. Let $\{G_n\}$ be the Zassenhaus filtration of G. If $[G : G_n] \gtrsim\!\!\!\!/\; n^d$ for all d, then $\gamma_G \gtrsim e^{\sqrt{n}}$.*

PROOF OF THEOREM A.1. The sequence $[G : G_n]$ was shown to be of superpolynomial growth in [**Gri89**], so Theorem A.7 yields the claimed lower bound; an elementary proof of this lower bound appears also in [**Gri84**].

For the upper bound, which is the main result of the present note, we invoke Proposition A.6 with $K = 1$, noting that $L_n \sim n^\alpha$ and $C_{L_n} \leq 4^{L_n}$, to obtain $\gamma_\omega \precsim (4{\cdot}2{\cdot}\gamma(1))^{n^\alpha} \sim e^{n^\alpha}$. □

A.5. Conclusion

The main fact used in the proof of Theorem A.1 is the existence of minimal forms given by Lemma A.4, coming from the natural map $\langle a\rangle * V \twoheadrightarrow G$. One can impose stronger conditions on minimal forms, such as 'not containing *dada* as a subword', coming from an explicit recursive presentation of G [**Lys85**]. Tighter upper bounds result from such considerations. Yurij Leonov [**Leo98b**] recently obtained improvements on the lower bound of Theorem A.1.

I wish to thank Robyn Curtis, Igor Lysënok and Pierre de la Harpe for having patiently listened to preliminary—and incorrect—proofs of these results, and also of course Rostislav Grigorchuk without whom these results couldn't even have existed.

ANNEXE B

Lower Bounds on the Growth of a Group acting on the Binary Rooted Tree

Laurent Bartholdi

B.1. Introduction

The growth of finitely generated groups, in relation with properties of Riemannian manifolds and coverings, was studied since the 1950's in the former Ussr [**Sva55**] and in the 1960's in the West [**Mil68a**]: let the finitely generated group G be the fundamental group of a compact CW-complex K with a suitable metric. Then the growth of G is equivalent to the growth of the universal cover $\tilde{K}$. There are well-known classes of groups of polynomial growth: abelian (K a torus, $\tilde{K}$ Euclidean space), and more generally virtually nilpotent groups (as was observed by John Milnor and Joseph Wolf; Mikhael Gromov proved conversely that all polynomial-growth groups are virtually nilpotent [**Gro81a**]); and classes of exponential growth: non-virtually-nilpotent linear [**Tit72**] or non-elementary hyperbolic [**GH90**] groups (K a negatively curved complex). John Milnor asked whether there exist finitely generated groups with growth greater than polynomial but less than exponential [**Mil68b**]. The first example of such a group, of intermediate growth, was discovered by Rostislav Grigorchuk, see [**Gri83, Gri84, Gri91**]. He showed that the growth functions γ of all of his groups satisfy

$$e^{\sqrt{n}} \precsim \gamma(n),$$

and that the growth of the main example of the construction, originally introduced in [**Gri80a**] as a solution to the general Burnside problem and called the "first Grigorchuk group" in [**Har00**], satisfies

$$\gamma(n) \precsim e^{n^\beta}, \qquad \beta = \log_{32}(31) \approx 0.991.$$

The author obtained in [**Bar98**] the following improvement on the upper bound: Let η be the real root of the polynomial X^3+X^2+X-2, and set $\beta' = \log(2)/\log(2/\eta) \approx 0.767$. Then the growth γ of Grigorchuk's group satisfies

$$\gamma(n) \precsim e^{n^{\beta'}}.$$

Recently, the same result was rediscovered and extended to the whole class of groups defined in [**Gri84**] by Roman Muchnik and Igor Pak [**MP99**], improving on the estimates in [**Gri84**].

We mentioned that the groups of polynomial growth are precisely the virtually nilpotent groups. Grigorchuk proved in [**Gri89**] that among residually-p groups, non-virtually-nilpotent groups have growth at least $e^{\sqrt{n}}$, and Alex Lubotzky and Avinoam Mann obtained the same result for residually-nilpotent groups—see [**BG00a**] for more. It is not known whether there exist groups of growth strictly between polynomial and $e^{\sqrt{n}}$.

A remaining outstanding problem in the theory of growth of groups is the question, raised by Grigorchuk in [**Gri84**], of the existence of groups with growth exactly $e^{\sqrt{n}}$. Such groups would have interesting extremal properties, for instance being of finite width [**BG00a**]. He conjectured in [**Gri91**, Conjecture 8.5] that his first group has this property, because of its finiteness of width and its just-infiniteness. However, in this paper we disprove that conjecture with the following result:

THEOREM B.1. *The growth of Grigorchuk's first 2-group G satisfies*

$$\gamma(n) \gtrsim e^{n^\alpha},$$

where $\alpha = 0.5157$.

The approach used in this paper to obtain lower bounds on the growth was suggested, with small variations, by Yuriĭ Leonov [**Leo98a**], where he announced $\gamma(n) \gtrsim e^{n^\beta}$ for $\beta = \log_{87/22}(2) \cong 0.504$. I wish to thank Yuriĭ for introducing me to this question, and Slava Grigorchuk and Pierre de la Harpe for their interest.

B.2. Growth of Groups

Let G be a group generated as a monoid by a finite set S. A *weight* on (G, S) is a function $\omega : S \to \mathbb{R}_+^*$. It induces a norm (again called a *weight*) ∂_ω on S^*, the free monoid on S, by setting $\partial_\omega(s_1 \dots s_n) = \omega(s_1) + \cdots + \omega(s_n)$, and a weight (again written ∂_ω) on G by

$$\partial_\omega : \begin{cases} G \to \mathbb{R}_+ \\ g \mapsto \min\{\omega(s_1) + \cdots + \omega(s_n) |\, s_1 \cdots s_n =_G g\}. \end{cases}$$

The important properties of the weight ∂_ω are its *submultiplicativity*: $\partial_\omega(gh) \leq \partial_\omega(g) + \partial_\omega(h)$ and its *properness*: $|\{g \in G |\, \partial_\omega(g) \leq n\}| < \infty$ for all $n \in \mathbb{N}$.

A *minimal form* of $g \in G$ is a representation of g as a word of minimal weight over S. If a minimal form has been fixed, it will be denoted by $\overline{g}$. The *growth* of G with respect to ω is then

$$\gamma_\omega : \begin{cases} \mathbb{R}_+ \to \mathbb{R}_+ \\ n \mapsto \#\{g \in G |\, \partial_\omega(g) \leq n\}. \end{cases}$$

Alternatively, S can altogether be suppressed from the definition, and a weight can be defined as a function $\omega : G \to \mathbb{R}_+$ that has finite support generating G as a monoid.

The function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is *dominated* by $\delta : \mathbb{R}_+ \to \mathbb{R}_+$, written $\gamma \precsim \delta$, if there is a constant $C \in \mathbb{R}_+$ such that $\gamma(n) \leq \delta(Cn)$ for all $n \in \mathbb{R}_+$. Two functions $\gamma, \delta : \mathbb{R}_+ \to \mathbb{R}_+$ are *equivalent*, written $\gamma \sim \delta$, if $\gamma \precsim \delta$ and $\delta \precsim \gamma$.

The following lemmata are well known:

LEMMA B.2. *Let S and S' be two finite generating sets for the group G, and let ω and ω' be weights on (G, S) and (G, S') respectively. Then $\gamma_\omega \sim \gamma_{\omega'}$.*

PROOF. Without loss of generality we may suppose $1 \notin S$, so $\partial_\omega(s) \neq 0$ for any $s \in S$. Let $C = \max_{s \in S} \partial_{\omega'}(s)/\partial_\omega(s)$. Then $\partial_{\omega'}(g) \leq C\partial_\omega(g)$ for all $g \in G$, and thus $\gamma_\omega(n) \leq \gamma_{\omega'}(Cn)$, from which $\gamma_\omega \precsim \gamma_{\omega'}$. The opposite relation holds by symmetry. □

In particular, for fixed S, any weight ∂_ω is equivalent to the length-weight $|g| = \min\{n | g = s_1 \dots s_n,\, s_i \in S\}$.

The *growth type* of a finitely generated group G is the $\sim$-equivalence class containing its growth functions; it will be denoted by γ_G.

Note that all exponential functions b^n $(b > 1)$ are equivalent, and polynomial functions of different degrees are inequivalent; the same holds for the subexponential functions e^{n^α}. We have

$$0 \precnsim n \precnsim n^2 \precnsim \cdots \precnsim e^{n^\alpha} \precnsim e^{n^\beta} \precnsim \cdots \precnsim e^n \qquad \text{for } 0 < \alpha < \beta < 1.$$

Note also that the ordering $\precsim$ is not linear. Actually, Slava Grigorchuk showed in his pioneering paper [**Gri84**, Theorem 7.2] that $\precsim$ admits chains and antichains with the cardinality of the continuum.

LEMMA B.3. *Let G be a finitely generated group. Then $\gamma_G \precsim e^n$.*

PROOF. Choose for G a finite generating set S, and define the weight ω by $\omega(s) = 1$ for all $s \in S$. Then $\gamma_\omega(n) \leq |S|^n$ for all n, so $\gamma_G \precsim e^n$. $\square$

If there is a $d \in \mathbb{N}$ such that $\gamma_G \precsim n^d$, the group G is of *polynomial growth* of degree at most d; if $\gamma_G \sim e^n$, then G is of *exponential growth*; otherwise G is of *intermediate growth*. As was already mentioned, the existence of groups of intermediate growth was first shown by Grigorchuk [**Gri83**].

LEMMA B.4. *Let $H < G$ be an index-N subgroup inclusion, let ω be a weight on G with growth function γ_ω, and let γ_ω^H denote the restriction of γ_ω to H:*

$$\gamma_\omega^H(n) = \#\{g \in H \,|\, \partial_\omega(g) \leq n\}.$$

Then there is a constant K such that

$$\gamma_\omega(n-K) \leq N\gamma_\omega^H(n) \leq \gamma_\omega(n+K)$$

holds for all n.

PROOF. Let T be a transversal of H in G, and set $K = \max_{t\in T} \partial_\omega(t)$. For every $g \in G$ of weight at most $n-K$, there is a unique $t \in T$ with $gt \in H$, and $\partial_\omega(gt) \leq n$. The map $\{G \to H,\, g \mapsto gt\}$ is N-to-1, so its restriction to the set of elements of weight at most $n-K$ is at most N-to-1, proving the first inequality.

Considering all $h \in H$ of weight at most n and all $t \in T$, we have $N\gamma_\omega^H(n)$ distinct elements $ht \in G$, with $\partial_\omega(ht) \leq n+K$. This proves the second inequality. $\square$

B.3. The Grigorchuk 2-Group

Let Σ^* be the set of finite sequences over the alphabet $\Sigma = \{0,1\}$. For $x \in \Sigma$ set $\overline{x} = 1-x$. Define recursively the following length-preserving permutations of Σ^*:

$$\begin{aligned} &a(x\sigma) = \overline{x}\sigma; \\ b(0\sigma) &= 0a(\sigma), & b(1\sigma) &= 1c(\sigma); \\ c(0\sigma) &= 0a(\sigma), & c(1\sigma) &= 1d(\sigma); \\ d(0\sigma) &= 0\sigma, & d(1\sigma) &= 1b(\sigma). \end{aligned}$$

Then G, the first Grigorchuk 2-group [**Gri80a**], is the group generated by $S = \{a,b,c,d\}$. It is readily checked that these generators are of order 2 and that $V = \{1,b,c,d\}$ is a Klein group.

Let $H = V^G$ be the normal closure of V in G. It is of index 2 in G and preserves the first letter of sequences; i.e. $H(x\Sigma^*) \subset x\Sigma^*$ for all $x \in \Sigma$. There is an injective homomorphism $\psi : H \to G \times G$, written $g \mapsto (g_0, g_1)$, defined by $0g_0(\sigma) = g(0\sigma)$ and $1g_1(\sigma) = g(1\sigma)$. As $H = \langle b,c,d,b^a,c^a,d^a\rangle$, we can write ψ explicitly as

$$\psi : \begin{cases} b \mapsto (a,c), & b^a \mapsto (c,a) \\ c \mapsto (a,d), & c^a \mapsto (d,a) \\ d \mapsto (1,b), & d^a \mapsto (b,1). \end{cases}$$

Let B be the normal closure of $\langle b\rangle$ in G. We shall use the following facts, whose proof appears for instance in [**Har00**, Section VIII.C]: B is of index 8 in G, and $\psi(H) = (B \times B) \rtimes \langle (a,1),(d,1)\rangle$ is of index 8 in $G \times G$, transversal to the order-8 dihedral group $\langle (a,d),(d,a)\rangle$.

B.4. The Growth of G

DEFINITION B.5. *A weight ω on (G,S) is* triangular *if for any ordering (x,y,z) of $\{b,c,d\}$ one has*

$$\omega(x) \leq \omega(y) + \omega(z).$$

We shall only consider triangular weights on G, and implicitly use the

LEMMA B.6. *Let ω be a triangular weight. Then every $g \in G$ admits a minimal form*

$$[*]a * a * a \cdots * a[*],$$

where $* \in \{b, c, d\}$ and the first and last $*$s are optional.

PROOF. Let w be a minimal form for $g \in G$. First, w may not contain two equal consecutive letters, since they could be cancelled, shortening the representation of g. Second, two unequal consecutive letters among $\{b, c, d\}$ can be replaced by the third, and as ω is triangular this operation will shorten the length of w while not enlarging its weight. Ultimately it will yield a word w' of the required form, also representing g, and with smaller or equal weight. It can then be chosen as a minimal form of g. $\square$

A lower estimate on the growth of G comes from the following proposition:

PROPOSITION B.7. *There are a weight ω on S and constants $K \geq 0, \eta < 4$ such that for all $h \in H$, writing $\psi(h) = (h_0, h_1)$, we have*

$$\partial_\omega(h) \leq \eta \max\{\partial_\omega(h_0), \partial_\omega(h_1)\} + K. \tag{22}$$

COROLLARY B.8. *The growth of G satisfies*

$$\gamma_G \gtrsim e^{n^\alpha},$$

where $\alpha = \log(2)/\log(\eta) \gneqq 0.5$.

PROOF. Applying Lemma B.4 (with constants K_1 and K_2) to the subgroup pairs $H < G$ and $\psi(H) < G \times G$, we obtain from the previous proposition

$$\frac{\gamma_\omega(\eta x + K + K_1)}{2} \geq \gamma_\omega^H(\eta x + K) \geq \gamma_{\omega\times\omega}^{\psi(H)}(x) \geq \frac{\gamma_\omega(x - K_2)^2}{8}$$

for all $x \geq K_2$. At the cost of increasing K, we rewrite it as $\gamma_\omega(\eta x + K) \geq \gamma_\omega(x)^2/4$, whence by iteration

$$\gamma_\omega\left(\eta^m x + \frac{\eta^m - 1}{\eta - 1}K\right) \geq \frac{\gamma_\omega(x)^{2^m}}{4^{2^m-1}}$$

for all integers $m \geq 1$ and $x \in \mathbb{R}_+$. Let $L \in \mathbb{R}_+$ be such that $\gamma(L) > 4$. Now given $n \in \mathbb{R}_+$, let $m \in \mathbb{N}$ be maximal such that $x := \eta^{-m}(n - K(\eta^m - 1)/(\eta - 1))$ be at least equal to L. We then have $\gamma_\omega(n) \geq (\gamma_\omega(x)/4)^{2^m} \geq (\gamma_\omega(L)/4)^{2^m}$, and since $m \approx \log(n)/\log(\eta)$, the corollary follows. $\square$

Note that it is easy to prove Proposition B.7 with the constant $\eta = 4$. Then by Corollary B.8 we would obtain $\gamma_G \gtrsim e^{\sqrt{n}}$.

PROOF OF PROPOSITION B.7 FOR $\eta = 4$. We reproduce, in a slightly different language, part of the proof of Theorem 3.2 in [**Gri84**]. Consider the weight $\omega(s) = 1$ for all $s \in S = \{a, b, c, d\}$ giving $\partial_\omega(g) = |g|$, and fix for every $g \in G$ a minimal form $\overline{g} \in S^*$. Let $\mathcal{M}$ be the set of these minimal forms, and define two maps on $\mathcal{M}$, induced by the letter-substitutions

$$\sigma : \begin{cases} \mathcal{M} \to \mathcal{M} \cap H \\ a \mapsto c^a, \\ b \mapsto d, \quad d \mapsto c, \quad c \mapsto b; \end{cases} \qquad \tau : \begin{cases} \mathcal{M} \to \mathcal{M} \\ a \mapsto d, \quad b \mapsto 1, \\ d \mapsto a, \quad c \mapsto a. \end{cases}$$

Now τ induces a group homomorphism in G, because $\tau = \rho\pi$, where π is projection on $G/B = \langle a, d\rangle \cong D_8$ and ρ is the automorphism of $\langle a, d\rangle$ swapping a and d. Since for any $g \in \mathcal{M}$, viewing ψ as a word-substitution, we have

$$\psi(\sigma(g)) = (\tau(g), g)$$

and ψ is injective, σ induces a group homomorphism from G to H. This result was already shown by Igor Lysionok [**Lys85**].

Now in G we have $|\tau(g)| \leq 4$, because $\tau(G) \cong D_8$, and $|\sigma(g)| \leq 2|g|+1$. For any $h \in H$, with $\psi(h) = (h_0, h_1)$, we have

$$h = a\sigma(h_0)a\sigma(\tau(h_0)^{-1}h_1);$$

indeed applying ψ to the right-hand side gives, for some $* \in G$,

$$(h_0, \tau(h_0))\big(\tau(\tau(h_0)^{-1}h_1), \tau(h_0)^{-1}h_1\big) = (h_0\tau(*), h_1) = (h_0, h_1),$$

because $(\tau(G), 1) \cap \psi(H) = (1, 1)$. Therefore we have

$$\begin{aligned} |h| &\leq |a| + |\sigma(h_0)| + |a| + |\sigma\tau(*)| + |\sigma(h_1)| \\ &\leq 1 + (2|h_0|+1) + 1 + 8 + (2|h_1|+1) \leq 4\max\{|h_0|, |h_1|\} + 12. \end{aligned}$$

□

B.5. Finite Transducers

We prove Proposition B.7 by constructing a "finite state automaton" Γ that, given a pair (h_0, h_1) of words determining an element in $\psi(H)$, constructs a preimage $h \in H$ satisfying (22).

The construction of Γ proceeds in four steps. In this section we describe transducers in general. In Section B.6 we describe sufficient conditions the automaton should satisfy in order to satisfy (22). In Section B.7 we explain how a computer was used to construct an appropriate Γ. Finally in Section B.8 we describe this transducer completely.

Definition B.9 (Padding). *Let S be an alphabet, and $\wp$ be a symbol not in S. The larger alphabet $\widehat{S} = S \cup \{\wp\}$ is called the associated* padded alphabet*.*

Given a word $u \in S^*$, the infinite word $\widehat{u} = u\wp^\infty$ is called the associated padded word*.*

If ω is a weight on S, we extend it to a weight on $\widehat{S}$ by $\omega(\wp) = 0$. It then follows that padding preserves a word's weight.

Padding is an useful technique in constructing transformation devices in that it avoids all 'end-of-input' situations; namely, rather than programming devices to handle in a special way an end-of-input, one adds a special symbol with precisely that meaning, and one may treat it as any other input symbol.

This is particularly useful if there is more than one input, and one of the input ends before the others.

Definition B.10 (Graphs). *A* graph *is a pair $\Gamma = (V, E)$ of sets, called respectively* vertices *and* edges*, with maps $\alpha, \omega : E \to V$ giving the* head *and* tail *of edges. The graph is* rooted *if a distinguished vertex $*$ has been fixed. It is* labeled *by $\mathcal{L}$ if there is an additional map $\lambda : E \to \mathcal{L}$. It is* finite *if V and E are finite.*

Let S be an alphabet. A 2-to-1 transducer *is a rooted graph Γ labeled by $\mathcal{L} = \widehat{S}^* \sqcup (\widehat{S}^* \times \widehat{S}^*)$. Traditionally vertices are called* states *and edges are called* transitions*. Edges labeled in $\widehat{S}^*$ are called* output transitions *and edges labeled in $\widehat{S}^* \times \widehat{S}^*$ are called* input transitions*.*

A path *π in a rooted graph Γ is a finite or infinite sequence $\pi = (e_1, e_2, \dots)$ of edges in Γ with $\alpha(e_1) = *$ and $\omega(e_i) = \alpha(e_{i+1})$ for all $i \geq 1$. Given a path π, its* input label *is $i(\pi) = (i(\pi)_0, i(\pi)_1) = \lambda(e_{i_1})\lambda(e_{i_2})\dots$, where $i_1 < i_2 < \dots$ is the set of indices of input transitions; its output label $o(\pi)$ is defined similarly.*

A weight $\partial\omega$ on S^* is extended to a weight, again written $\partial\omega$, on $S^* \times S^*$ by

$$\partial_\omega(u_0, u_1) = \partial_\omega(u_0) + \partial_\omega(u_1).$$

A transducer Γ operates as follows, when possible: given a pair of words (u_0, u_1) over S, let π be a path in Γ with $i(\pi) = (\widehat{u_0}, \widehat{u_1})$. Assume $o(\pi)$ is a padded word, say $o(\pi) = \widehat{v}$ for some $v \in S^*$. Then the output of Γ is v.

If there exists an appropriate path π, the pair (u_0, u_1) is called *acceptable*. If moreover the path π is unique for all acceptable pairs, the transducer is called *deterministic*.

The transducer should informally be understood as follows: it has some input; at each node it decides whether it wants to read more input or is ready for output; after having exhausted its input it will read $(\wp,\wp)$ and output $\wp$ symbols.

A *loop* ℓ in a graph Γ is a sequence $(\ell_1,\dots,\ell_n)$ of edges with $\omega(\ell_i)=\alpha(\ell_{i+1})$ for $i=1,\dots,n$ and $\omega(\ell_n)=\alpha(\ell_1)$. It is *injective* if all ℓ_i are distinct. Clearly in a finite graph there is only a finite number of injective loops. The following lemma's conditions can then be checked in a finite time.

LEMMA B.11. *Let Γ be a finite transducer over S, and let ω be a weight on words over S. Let η be such that for all injective loops ℓ in Γ the following holds:*

$$\partial_\omega o(\ell) < \frac{\eta}{2}(\partial_\omega i(\ell)_0 + \partial_\omega i(\ell)_1). \tag{23}$$

Then there is a constant K such that, for every acceptable input pair (u_0,u_1), there is an output word v with

$$\partial_\omega v \le \eta \max\{\partial_\omega u_0, \partial_\omega u_1\} + K.$$

PROOF. Let K be the total weight of all output transitions: $K=\sum_{e\text{: output}}\partial_\omega(\lambda(e))$. We prove by induction on $\partial_\omega u_0+\partial_\omega u_1$ that

$$\partial_\omega v \le \frac{\eta}{2}(\partial_\omega u_0 + \partial_\omega u_1) + K. \tag{24}$$

Let $\pi=(e_1,e_2,\dots)$ be a path in Γ, with $i(\pi)=(\widehat{u}_0,\widehat{u}_1)$ and $o(\pi)=\widehat{v}$. Since the pair $(\widehat{u}_0,\widehat{u}_1)$ is padded, there is only a finite number of input transitions e_i in π with $\partial_\omega(\lambda(e_i))\neq 0$.

If all these e_i are distinct, then $\partial_\omega(o(e))\le K$ and v satisfies 24.

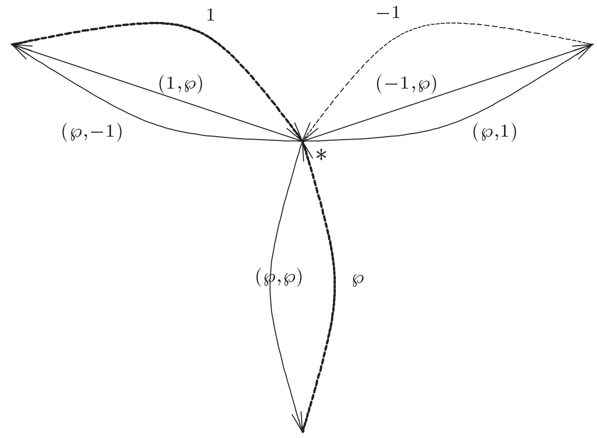
If, say, $e_i=e_j$ for some $i<j$, then $(e_i,\dots,e_{j-1})$ is a loop of non-zero input weight, and it contains an injective subloop $\ell=(e_{i'},\dots,e_{j'-1})$ with $i\le i'<j'\le j$, which satisfies (23). Let e' be the path e from which ℓ has been deleted. Since e' has smaller input weight than e, we may suppose by induction that e' satisfies (24), and we have

$$\begin{aligned}\partial_\omega v &= \partial_\omega(o(e')) + \partial_\omega(o(\ell))\\ &\le \frac{\eta}{2}(\partial_\omega i(e')_0 + \partial_\omega i(e')_1) + K + \frac{\eta}{2}(\partial_\omega i(\ell)_0 + \partial_\omega i(\ell)_1)\\ &= \frac{\eta}{2}(\partial_\omega u_0 + \partial_\omega u_1) + K.\end{aligned}$$

The proof is finished by noting that $X+Y\le 2\max\{X,Y\}$ for all X,Y. □

As an simple example of transducer, we present the *subtractor*. Its alphabet is $S=\{1,-1\}$; a word $w\in S^*$ has *value* $\#w=n$ obtained from computing in $\mathbb{Z}$. The following transducer, when presented (u_0,u_1) as input, produces a word v with $\#v=\#u_0-\#u_1$. Clearly all injective loops in v satisfy (23) with $\eta=2$. Therefore Lemma B.11 is a rephrasing of the inequality $|x-y|\le 2\max\{|x|,|y|\}$ for $x,y\in\mathbb{Z}$.

The subtractor transducer has 4 states, 5 input transitions (represented as solid arrows) and 3 output transitions (represented as dashed arrows).



B.5.1. A transducer for $1 \times B$. As a concrete example, we describe a transducer Δ accepting as input all pairs $(\wp^\infty, u\wp^\infty)$ where u is a word on $\{a, b, c, d\}$ representing an element in B (the normal closure of b in G), and is of the form

$$x_1 a x_2 a \dots x_n a,$$

with $x_i \in \{b, c, d\}$ and without consecutive d's (this condition can be imposed because the transformation $x_i adad \Rightarrow (x_i d)ada$ reduces the number of consecutive d's in u without changing its value in B). The transducer produces as output a word $v\wp^\infty$ with $\psi(v) = (1, u)$.

It is described in Figure B.1. Each box represents two vertices in Δ, with an output edge, labeled by the box's contents, connecting the first vertex to the second. All other edges are input transitions, start at the second vertex in the source box, and end at the first vertex in the destination box. Some edges have double arrows ($\leftrightarrow$) and represent two input edges in Δ. Note that all paths in Δ are alternating between input and output edges. The root is the second vertex of the box marked $*$.

The transitions are represented with a word u. Their label is $(\wp, u)$. Additionally, there is an input transition labeled $(\wp, \wp)$ looping at the box labeled $\wp$, from which no transition escapes.

For example, let us study the behaviour of Δ for the input word $u = (ca)^4$. In Δ, the corresponding path goes clockwise through the four bottom nodes and ends at $*$. The corresponding output is read from the boxes traversed, namely $v = d^{aca} d^{acacaca} d^{caca} d$. One has $\psi(v) = (1, b^a b^{ada} b^{da} b) = (1, (ac)^4) = (1, u)$, because $(ac)^8 = 1$.

Note that for all paths in Δ the input and output transitions are followed in alternance. We choose for S the weight $\omega(s) \equiv 1$. Then the input transitions' labels all have weight at least 2, and the output transitions have weight at most 15. It then follows that $|v| \le 15/2|u|$, and Lemma B.11 holds for $\eta = 15/2$. By careful examination, one may check that all injective loops in Δ actually satisfy (23) for $\eta = 4$; this provides an alternate proof of Proposition B.7 for $\eta = 4$.

B.6. Description of Γ

We now describe in closer detail the transducer Γ that shall satisfy the conditions of Lemma B.11. First we fix a minimal form $\overline{g}$ for every $g \in G$, and denote by $\mathcal{M} \subset S^*$ this set of minimal forms. In Section B.7 we shall describe a computer-aided process that constructs a triangular weight $\omega : \{a, b, c, d\} \to \mathbb{R}_+^*$ and the finite oriented graph (also denoted Γ) representing the transducer. Let us first highlight the important properties of Γ:

- It is finite.

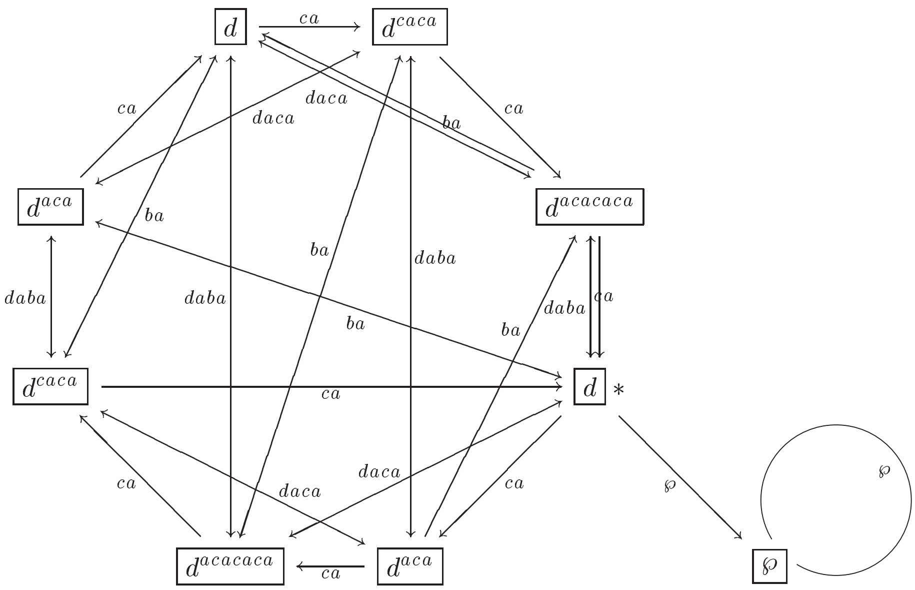


Figure B.1: The transducer Δ for $1 \times B$

- $V(\Gamma) \subset \mathcal{M} \times \mathcal{M}$, i.e. each vertex is identified with a pair of words (u_0, u_1) from the set of minimal forms. Using this identification, $V(\Gamma)$ is also viewed as a subset of $G \times G$.

  The vertex set of Γ corresponds to the machine's "memory buffer": roughly, when it reads a symbol pair, it adds it at the right of its buffer, while when it outputs a symbol, it deletes some part on the left of its buffer.
- $* = (\lambda, \lambda)$ is a vertex, where λ is the empty word, representing the unit element in G.
- At each node, either all edges are input or all are output transitions. Accordingly, we say the node itself is of type *input* or *output* according to its outgoing transitions. An input node has 9 input transitions, namely with all labels of the form (xa, ya) for all $x, y \in \{b, c, d\}$. An output node has a single outbound transition, with a label in $\mathcal{M}$.
- An input transition labeled (xa, ya) at a vertex (u_0, u_1) ends at $(\overline{u_0 xa}, \overline{u_1 ya})$. An output transition labeled v at a vertex (u_0, u_1) ends at $(\overline{v_0 u_0}, \overline{v_1 u_1})$.
- At every input vertex $(u_0, u_1) \in \psi(H)$, and for every $u \in \mathcal{M} \cap B$ of length of most 8, there is an input transition labeled (λ, u) from (u_0, u_1) to a new vertex, directly followed by an output transition to (λ, λ) labeled by some v with $\psi(v) = (u_0, u_1 u)$.
- Except for these extra transitions, Γ satisfies Lemma B.11 with the same constant η.

For now, we suppose such a graph exists, and prove Proposition B.7 under that assumption.

Proof of Proposition B.7. Let $(u_0, u_1) \in \psi(H)$ be a pair of words in minimal form representing (h_0, h_1). For some $\eta < 4$, K and ω to determine, we are to construct a word v such that $\psi(v) = (u_0, u_1)$ and $\partial_\omega(v) \leq \eta \max\{\partial_\omega(u_0), \partial_\omega(u_1)\} + K$.

First we may suppose that neither u_0 nor u_1 starts by 'a'; indeed we may add b's at their beginning (remembering that $(b, 1)$ and $(1, b)$ are in $\psi(H)$) and construct a word v' such that $\psi(v') = (b^i u_0, b^j u_1)$ with $i, j \in \{0, 1\}$. The word $v = (ada)^i d^j v'$ then satisfies

$\psi(v) = (u_0, u_1)$ and is of weight at most $\partial_\omega(d) + \partial_\omega(ada)$ more than v', a fact that can be coped with by increasing the constant K.

Second, we may suppose that u_0 is shorter than u_1; indeed if u_1 is shorter, we may apply the Proposition to the pair (u_1, u_0), obtaining a word v'. The word $v = av'a$ satisfies (22) if one increases K by $2\omega(a)$.

Third, we may suppose that the lengths of u_0 and u_1 differ by at most at most 8. Indeed u_0 can be extended by copies of $(ad)^4$ which is trivial in G.

The first condition is necessary because all input transitions in Γ are of the form (xa, ya) with $x, y \in \{b, c, d\}$. The second and third are needed because the only input transitions (w_0, w_1) with $|w_0| \neq |w_1|$ have $|w_0|$ shorter, and their length-difference is at most 8.

Conversely, if (u_0, u_1) satisfy these three conditions, there is a path in Γ with $i(\pi) = (\widehat{u}_0, \widehat{u}_1)$: one starts at $*$, follows input transitions by taking 2 letters from u_0 and u_1 at a time, follows output transitions when needed, follows a (λ, u) transition if u_1 is longer than u_0, and finally loops at $*$ inputting and outputting $\wp$'s.

Now Γ satisfies Lemma B.11 for the constant η. This implies that $v = o(\pi)$ represents an element $h \in H$ satisfying (22), because $\partial_\omega(h) \leq \partial_\omega(v)$ for any word v representing an element h. □

B.7. Construction of Γ

In this section we explain how a computer was programmed to construct the graph Γ described in the previous section. The construction itself involved a large amount of experimenting, to find adequate values for the constants described below.

First an arbitrary triangular weight is selected, for instance $\omega(s) \equiv 1$ (in practice, values between 0.9 and 1.1 work well); a tiny δ (about 0.01), a largish η' (around 4) and a limit-length N (around 20) are also chosen.

The *quality* of an output transition e labeled v from (u_0, u_1) to (u_0', u_1') is defined as

$$q(e) = \frac{\partial(u_0) + \partial(u_1) - \partial(u_0') - \partial(u_1')}{\partial(v)} + \delta\,|\partial(u_0) - \partial(u_1)| - \delta\,|\partial(u_0') - \partial(u_1')|\,.$$

We see edges as transitions. The more a transition decreases vertex weights (relatively to the output's weight), the higher its quality; and the more a transition balances the weight between its vertices, the higher its quality. This extra condition prevents the graph from becoming too large in the automatic construction process.

The graph Γ is now constructed iteratively. At all steps of the iteration we shall have a graph Γ satisfying all the required conditions, except that it will have "hanging edges", that is, edges not connected to any vertex. The purpose of the iteration process will then be to attach the hanging edges, either to existent vertices in Γ or to new ones.

The graph Γ starts with a single vertex, (λ, λ), and a hanging edge. Then, while there are hanging edges, the following is performed:

- At each node (u_0, u_1) with a hanging edge, all words $v \in \mathcal{M} \cap H$ of length at most N are tried; we write $\psi(v) = (v_0, v_1)$.
- If the quality of the contracting edge from (u_0, u_1) to $(\overline{v_0 u_0}, \overline{v_1 u_1})$ is at least $1/\eta'$, then the type of (u_0, u_1) is set to "output", and the hanging edge is replaced by an output transition from (u_0, u_1) to $(\overline{v_0 u_0}, \overline{v_1 u_1})$, labeled v.
- If there is no such output transition of sufficient quality, then the type of (u_0, u_1) is set to "input", and the hanging edge is replaced by 9 edges to all the $(\overline{u_0 xa}, \overline{u_1 ya})$ for all $x, y \in \{b, c, d\}$.

If the process stops, we have obtained a graph Γ satisfying all constraints, including Lemma B.11 for the constant η'. In fact, it may well satisfy Lemma B.11 for some smaller value of η: the triangular weight may be varied, and it may happen that all edges of low quality are surrounded by edges of high quality.

To make the constant η smaller, a second program was written. It takes as input the description of Γ, and adjusts slightly the triangular weight ω; then it searches in Γ all injective loops, and on each loop computes the weight of the input- and output-labels. If the adjustment decreases the maximal ratio of input-weight to output-weight, it is kept. Then another adjustment is tried, etc.

Finally the "special transitions" $(\wp^{|u|}, u)$ are added to Γ.

In actual computations, a graph Γ with 160 vertices (of which 12 are input states) was constructed; its complete description is given in the next section. An appropriate weight was then chosen to be

$$\omega(a) = 1, \quad \omega(b) = 3.33, \quad \omega(c) = 2.8, \quad \omega(d) = 1.06.$$

The computer program computed the input/output weight ratio for all the injective loops in Γ. Its maximal value is $\eta \approx 3.83414$.

In fine, the proof of Proposition B.7 relies on a finite, but tedious computation in the graph Γ. Though it could have extensively been checked by a human, we shall content ourselves with a check on one injective loop. Consider the input $((ca)^2, (ca)^2$. It corresponds to the injective loop

$$(\lambda, \lambda) \to (\mathtt{ca}, \mathtt{ca}) \to (\mathtt{caca}, \mathtt{caca}) \to (\lambda, \lambda)$$

with output label $\mathtt{adabacabacabadab} = d^a bc^a bc^a bd^a b$. The weight of the input is $2\omega(c)+2\omega(a) = 7.6$, and the weight of the output is $4\omega(b) + 2\omega(c) + 2\omega(d) + 8\omega(a) = 29.04$. The ratio of these numbers is about 3.82, less than η.

B.8. The Graph Γ

In this printout, the graph Γ satisfying the conditions of Lemma B.11 for the constant $\eta \approx 3.83414$ is described. The data are structured in "paragraphs", each paragraph corresponding to a state of type "input". Its nine successors are listed in the order $(da, da), (da, ca), \ldots, (ba, ba)$, and each of these successors is identified by its type: if it is of type "output", it is followed by the symbol

$\mathtt{OUTPUT}(w) \;\to\; (u_0, u_1)$

meaning the word w is output and the next state to be in is (u_0, u_1). If the successor is of type "input", it is followed by the symbol `INPUT` and its successors are described in another paragraph.

Input states are listed in lexicographical order, starting with pairs (u_0, u_1) of same length, then pairs with length-difference 1, then 2, etc. In the lexicographical order d comes before c which comes before b. The input states are (λ, λ), (da, da), (da, ca), (ca, da), (ca, ca), (a, λ), (da, d), (da, λ), (ca, λ), (ad, λ), (ada, λ) and (aca, λ).

first, the initial (and terminal) vertex:

```
(λ,λ):  INPUT → (da,da):  INPUT
                (da,ca):  INPUT
                (da,ba):  OUTPUT(daca) → (a,λ)
                (ca,da):  INPUT
                (ca,ca):  INPUT
                (ca,ba):  OUTPUT(daba) → (a,λ)
                (ba,da):  OUTPUT(adac) → (a,λ)
                (ba,ca):  OUTPUT(adab) → (a,λ)
                (ba,ba):  OUTPUT(daba) → (da,λ)
```

this finishes the description of (λ,λ)*. Now comes the description of those successors that are of type "input"*

```
(da,da):  INPUT → (dada,dada)=(adad,adad):  OUTPUT(acacacac) → (λ,λ)
```

this is the first loop reached by the algorithm: its input-label is $(dada, dada)$ *and its output-label is* $c^a cc^a c$*. It has an input-output ratio of* $\eta = 3.69$*. Note the simplification* $dada = adad$

```
                  (dada,daca)=(adad,daca):  OUTPUT(cacabaca) → (λ,λ)
                  (dada,daba)=(adad,daba):  OUTPUT(cacab) → (da,d)
                  (daca,dada)=(daca,adad):  OUTPUT(acacabac) → (λ,λ)
                  (daca,daca):  OUTPUT(cacabacabadab) → (λ,λ)
                  (daca,daba):  OUTPUT(acacabac) → (aca,λ)
                  (daba,dada)=(daba,adad):  OUTPUT(acacaba) → (da,d)
                  (daba,daca):  OUTPUT(cacabaca) → (aca,λ)
                  (daba,daba):  OUTPUT(acacabacacabaca) → (ad,λ)
(da,ca):  INPUT → (dada,cada)=(adad,cada):  OUTPUT(bacacaca) → (λ,λ)
                  (dada,caca)=(adad,caca):  OUTPUT(bacabaca) → (λ,λ)
                  (dada,caba)=(adad,caba):  OUTPUT(bacab) → (da,d)
                  (daca,cada):  OUTPUT(dacacabac) → (λ,λ)
```

here is another loop at (λ,λ)*: its input-label is* $(daca, cada)$ *and its output-label is* $dc^a cb^a c$*. It has an output-input ratio of* $\eta = 3.04$

```
                  (daca,caca):  OUTPUT(bacabacabadab) → (λ,λ)
                  (daca,caba):  OUTPUT(dacacabac) → (aca,λ)
                  (daba,cada):  OUTPUT(dacacaba) → (da,d)
                  (daba,caca):  OUTPUT(bacabaca) → (aca,λ)
                  (daba,caba):  OUTPUT(bacabacacabac) → (ad,λ)
(ca,da):  INPUT → (cada,dada)=(cada,adad):  OUTPUT(abacacac) → (λ,λ)
                  (cada,daca):  OUTPUT(adacacabaca) → (λ,λ)
                  (cada,daba):  OUTPUT(adacacab) → (da,d)
                  (caca,dada)=(caca,adad):  OUTPUT(abacabac) → (λ,λ)
                  (caca,daca):  OUTPUT(abacabacabadaba) → (λ,λ)
                  (caca,daba):  OUTPUT(abacabac) → (aca,λ)
                  (caba,dada)=(caba,adad):  OUTPUT(abacaba) → (da,d)
                  (caba,daca):  OUTPUT(adacacabaca) → (aca,λ)
                  (caba,daba):  OUTPUT(abacabacacabaca) → (ad,λ)
(ca,ca):  INPUT → (cada,cada):  OUTPUT(adabacacaca) → (λ,λ)
                  (cada,caca):  OUTPUT(adabacabaca) → (λ,λ)
                  (cada,caba):  OUTPUT(adabacab) → (da,d)
                  (caca,cada):  OUTPUT(dabacabac) → (λ,λ)
                  (caca,caca):  OUTPUT(adabacabacabadab) → (λ,λ)
                  (caca,caba):  OUTPUT(dabacabac) → (aca,λ)
                  (caba,cada):  OUTPUT(dabacaba) → (da,d)
                  (caba,caca):  OUTPUT(adabacabaca) → (aca,λ)
                  (caba,caba):  OUTPUT(ababadabacababadaba) → (ad,λ)
```

now come vertices whose lengths differ by 1*:*

```
(a,λ):  INPUT → (ada,da):  OUTPUT(caca) → (a,λ)
                (ada,ca):  OUTPUT(baca) → (a,λ)
                (ada,ba):  OUTPUT(b) → (da,da)
                (aca,da):  OUTPUT(caba) → (a,λ)
```

```
(aca,ca):  OUTPUT(baba) → (a,λ)
(aca,ba):  OUTPUT(b) → (ca,da)
(aba,da):  OUTPUT(caba) → (da,λ)
(aba,ca):  OUTPUT(baba) → (da,λ)
(aba,ba):  OUTPUT(cadab) → (a,λ)
```

---

in fact, the destination edge is (λ, a), but we list vertices with the longest word first. This has no incidence on the computations, since word reductions are similar when operating on (u_0, u_1) and ond (u_1, u_0)

```
(da,d):  INPUT → (dada,dda)=(adad,a):  OUTPUT(aca) → (ada,λ)
                 (dada,dca)=(adad,ba):  OUTPUT(daca) → (ada,λ)
                 (dada,dba)=(adad,ca):  OUTPUT(baca) → (ad,λ)
                 (daca,dda)=(daca,a):  OUTPUT(aca) → (aca,λ)
                 (daca,dca)=(daca,ba):  OUTPUT(daca) → (aca,λ)
                 (daca,dba)=(daca,ca):  OUTPUT(dacacabac) → (ad,λ)
                 (daba,dda)=(daba,a):  OUTPUT(acabadab) → (λ,λ)
                 (daba,dca)=(daba,ba):  OUTPUT(dacabadab) → (λ,λ)
                 (daba,dba)=(daba,ca):  OUTPUT(acabadab) → (aca,λ)
```

---

now come vertices whose lengths differ by 2:

```
(da,λ):  INPUT → (dada,da)=(adad,da):  OUTPUT(caca) → (ad,λ)
                 (dada,ca)=(adad,ca):  OUTPUT(baca) → (ad,λ)
                 (dada,ba)=(adad,ba):  OUTPUT(daca) → (ada,λ)
                 (daca,da):  OUTPUT(acacabac) → (ad,λ)
                 (daca,ca):  OUTPUT(dacacabac) → (ad,λ)
                 (daca,ba):  OUTPUT(daca) → (aca,λ)
                 (daba,da):  OUTPUT(acabadab) → (ada,λ)
                 (daba,ca):  OUTPUT(acabadab) → (aca,λ)
                 (daba,ba):  OUTPUT(dacabadab) → (λ,λ)
(ca,λ):  INPUT → (cada,da):  OUTPUT(adacaca) → (ad,λ)
                 (cada,ca):  OUTPUT(adabaca) → (ad,λ)
                 (cada,ba):  OUTPUT(daba) → (ada,λ)
                 (caca,da):  OUTPUT(abacabac) → (ad,λ)
                 (caca,ca):  OUTPUT(dabacabac) → (ad,λ)
                 (caca,ba):  OUTPUT(daba) → (aca,λ)
                 (caba,da):  OUTPUT(ababadab) → (ada,λ)
                 (caba,ca):  OUTPUT(ababadab) → (aca,λ)
                 (caba,ba):  OUTPUT(dababadab) → (λ,λ)
(ad,λ):  INPUT → (adda,da)=(λ,da):  INPUT
                 (adda,ca)=(λ,ca):  INPUT
                 (adda,ba)=(λ,ba):  OUTPUT(d) → (a,λ)
                 (adca,da)=(aba,da):  OUTPUT(caba) → (da,λ)
                 (adca,ca)=(aba,ca):  OUTPUT(baba) → (da,λ)
                 (adca,ba)=(aba,ba):  OUTPUT(cadab) → (a,λ)
                 (adba,da)=(aca,da):  OUTPUT(caba) → (a,λ)
                 (adba,ca)=(aca,ca):  OUTPUT(baba) → (a,λ
                 (adba,ba)=(aca,ba):  OUTPUT(b) → (ca,da)
```

---

finally some vertices whose lengths differ by 3:

```
(ada,λ):  INPUT → (adada,da)=(dad,da):  OUTPUT(caca) → (ada,λ)
                  (adada,ca)=(dad,ca):  OUTPUT(baca) → (ada,λ)
                  (adada,ba)=(dad,ba):  OUTPUT(daca) → (ad,λ)
                  (adaca,da):  OUTPUT(c) → (daca,a)
                  (adaca,ca):  OUTPUT(b) → (daca,a)
                  (adaca,ba):  OUTPUT(b) → (daca,da)
                  (adaba,da):  OUTPUT(c) → (daba,a)
                  (adaba,ca):  OUTPUT(b) → (daba,a)
                  (adaba,ba):  OUTPUT(b) → (daba,da)
(aca,λ):  INPUT → (acada,da):  OUTPUT(caba) → (ada,λ)
                  (acada,ca):  OUTPUT(baba) → (ada,λ)
                  (acada,ba):  OUTPUT(badacaca) → (ad,λ)
                  (acaca,da):  OUTPUT(caba) → (aca,λ)
                  (acaca,ca):  OUTPUT(baba) → (aca,λ)
```

```
(acaca,ba):  OUTPUT(b) → (caca,da)
(acaba,da):  OUTPUT(cabababadab) → (λ,λ)
(acaba,ca):  OUTPUT(babababadab) → (λ,λ)
(acaba,ba):  OUTPUT(b) → (caba,da)
```

ANNEXE C

Salem numbers and growth series of some hyperbolic graphs

Laurent Bartholdi and Tullio G. Ceccherini-Silberstein

C.1. Introduction

We consider the graphs $\mathcal{X}_{\ell,m}$ ($\ell, m \geq 3$) defined by Floyd and Plotnick in [**FP94**]. These graphs are ℓ-regular and are the 1-skeleton of a tessellation of the sphere (if $(\ell-2)(m-2) < 4$), the Euclidean plane (if $(\ell-2)(m-2) = 4$) or the hyperbolic plane (if $(\ell-2)(m-2) > 4$) by regular m-gons. These tessellations were studied by Coxeter [**Cox54**]. When $m = \ell = 4g$ with g at least two, $\mathcal{X}_{\ell,m}$ is the Cayley graph of the fundamental group $J_g = \pi_1(\Sigma_g)$ of an orientable closed surface Σ_g of genus g, with respect to the usual set of generators $S_g = \{a_1, b_1, \dots, a_g, b_g\}$:

$$J_g = \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] \right\rangle.$$

The growth series for J_g with respect to S_g, namely

$$F_g(X) = \sum_{s \in J_g} X^{|s|} = \sum_{n=0}^{\infty} f_n X^n,$$

where $|s| = \min\{t : s = s_1 \dots s_t, s_i \in S_g \cup S_g^{-1}\}$ denotes the word length of s with respect to S_g and $f_n = |\{s \in J_g : |s| = n\}|$, was computed by Cannon and Wagreich in [**Can83**] and [**CW92**] and shown to be rational, indeed

$$F_g(X) = \frac{1 + 2X + \cdots + 2X^{2g-1} + X^{2g}}{1 - (4g-2)X - \cdots - (4g-2)X^{2g-1} + X^{2g}};$$

moreover they showed that the denominator is a Salem polynomial. It was later shown by Floyd [**Flo92**] and Parry [**Par93**] that the denominators of the growth series of Coxeter groups are also Salem polynomials.

In [**FP87**] and [**FP94**], Floyd and Plotnick, among other things, extended the calculations of Cannon and Wagreich to the family $\mathcal{X}_{\ell,m}$. Fixing arbitrarily a base point $* \in V(\mathcal{X}_{\ell,m})$ and denoting by $|x|$ the graph distance between the vertices x and $*$, they obtained the following formulæ for the growth series $F_{\ell,m}(X) = \sum_{x \in V(\mathcal{X}_{\ell,m})} X^{|x|}$; for m even, say $m = 2w$:

$$F_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{w-1} + X^{w}}{1 - (\ell-2)X - \cdots - (\ell-2)X^{w-1} + X^{w}} \tag{25}$$

and, for m odd, say $m = 2w+1$:

$$F_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{w-1} + 4X^{w} + 2X^{w+1} + \cdots + 2X^{2w-1} + X^{2w}}{1 - (\ell-2)X - \cdots - (\ell-4)X^{w} - \cdots - (\ell-2)X^{2w-1} + X^{2w}}. \tag{26}$$

Our main result is the following:

THEOREM C.1. *The denominators of the growth series $F_{\ell,m}$ are reciprocal Salem polynomials. After simplification by $X+1$ in case $m \equiv 2 \mod 4$, they are irreducible, except for the following exceptional cases:*

(1) $\ell = 3$ *and* $m \equiv 2 \mod 6$*; there is a factor of* $X^2 - X + 1$*;*
(2) $\ell = 3$ *and* $m \equiv 3 \mod 6$*; there is a factor of* $X^2 + X + 1$*;*
(3) $\ell = 3$ *and* $m \equiv 5 \mod 20$*; there is a factor of* $X^4 - X^3 + X^2 - X + 1$*;*
(4) $\ell = 4$ *and* $m \equiv 3 \mod 8$*; there is a factor of* $X^2 + 1$*;*
(5) $\ell = 5$ *and* $m \equiv 3 \mod 12$*; there is a factor of* $X^2 - X + 1$*.*

This theorem extends the results in Section 4 of [**FP94**] which show, for even m, that the denominator of $F_{\ell,m}$ is a product of an irreducible Salem polynomial and distinct cyclotomic polynomials.

COROLLARY C.2. *The growth rates of the graphs $\mathcal{X}_{\ell,m}$ are Salem numbers.*

We thus obtain more precise information about the growth coefficients:

COROLLARY C.3. *If $F_{\ell,m}(X) = \sum_{n\geq 0} f_n X^n$, then there exist constants $K > 0$, $\lambda > 1$ and $R > 0$ such that*

$$K\lambda^n - R < f_n < K\lambda^n + R$$

holds for all n. Moreover λ is a Salem number.

This improves, for the graphs $\mathcal{X}_{\ell,m}$, on a general result by Coornaert [**Coo93**] for non-elementary hyperbolic groups, which asserts that there exist constants $\lambda > 1$ and $0 < K_1 < K_2$ such that

$$K_1\lambda^n < f_n < K_2\lambda^n$$

for all n.

Note that Corollary C.3 does not hold for a general presentation of a hyperbolic group, nor even of a virtually free group. Consider for instance the modular group $PSL_2(\mathbb{Z}) = \langle a, b |\, a^2 = b^3 = 1\rangle$. The growth coefficients f_n satisfy

$$K\phi^n - 2 < f_n < K\phi^n + 2 \qquad \text{with } K = \frac{3+\sqrt{5}}{\sqrt{5}}, \phi = \frac{1+\sqrt{5}}{2}$$

for the generating set $\{a, ab, (ab)^{-1}\}$ (note that ϕ is a Salem number), but for the generating set $\{a, b, b^{-1}\}$ are

$$f_n = \begin{cases} 2\cdot\sqrt{2}^n & \text{if } n \text{ is even,} \\ 3/\sqrt{2}\cdot\sqrt{2}^n & \text{if } n \text{ is odd.} \end{cases}$$

These computations are due to Machì; see [**Har00**, VI.7]. It follows that Corollary C.3 does not extend to arbitrary hyperbolic group presentations (indeed, $\sqrt{2}$ is not even a Salem number).

The authors express their thanks to Pierre de la Harpe, Alexander Borisov and Kurt Foster for their interest and generous contribution.

C.2. Salem Polynomials

We recall a few facts on Salem polynomials; one might also consult [**BDG⁺92**, § 5.2] or the original paper [**Sal45**]. A polynomial $f(X) = f_0 + f_1X + \cdots + f_nX^n$ ($f_n \neq 0$) is *reciprocal* if $f_i = f_{n-i}$ for all $i = 0, 1, \ldots, n$; equivalently if $X^n f(X^{-1}) = f(X)$.

A polynomial $f(X) \in \mathbb{Z}[X]$ is a *Salem polynomial* if it is monic, admits exactly one root λ of modulus $|\lambda| > 1$, and this root is simple; this λ is then necessarily real. If moreover f is reciprocal, then $1/\lambda$ is also a root of f and these are the only roots of f not on the unit circle.

If f is a reciprocal polynomial of odd degree, then $f(-1) = 0$, so $X+1$ divides f. There is therefore no real limitation in considering reciprocal Salem polynomials of even degree.

If f is a Salem polynomial and g is a cyclotomic polynomial, then fg is again a Salem polynomial, and reciprocally any Salem polynomial can be factored as a product of an irreducible Salem polynomial and cyclotomic polynomials.

A real number λ is called a *Salem number* if $\lambda > 1$, λ is an algebraic integer, and all its conjugates except $\lambda^{\pm 1}$ have absolute value 1; equivalently if $\lambda > 1$ is the root of a Salem polynomial.

For instance, the reciprocal Salem polynomials of degree 2 are the $X^2 - aX + 1$ for all $a \in \mathbb{Z}$ with $a \geq 3$. The corresponding Salem numbers are the $(a + \sqrt{a^2-4})/2$. The Salem polynomials of degree 2 are all the $X^2 + aX + b$ subject to $1 + c < b$.

Denoting by

$$\Phi(z) = \frac{z-i}{z+i}$$

the Cayley transform [**Rud91**, 13.17] mapping the extended real axis $\mathbb{R} \cup \{\infty\}$ onto the unit circle $\mathbb{T}$, we give the following

Definition C.4. *Let f be a polynomial of degree n. Its* Cayley transform *is the polynomial*

$$C(f)(X) = (X+i)^n f(\Phi(X)).$$

Note that if f is real and reciprocal, its Cayley transform will again be real.

The proof of the following characterization of reciprocal Salem polynomials is straightforward:

Theorem C.5. *Let f be a monic integral reciprocal polynomial of degree n. Then f is a Salem polynomial if and only if the polynomial $C(f)$ has exactly $n-2$ real roots (its last two roots are then complex conjugate).*

C.3. The Denominators of the Growth Series $F_{\ell,m}$

The objective of this section is to prove that the denominator of $F_{\ell,m}$ is (after a possible division by $X+1$ depending on the parity of its degree m) a reciprocal Salem polynomial. For this purpose define the following auxiliary polynomials, for $a \in \mathbb{Z}$ and $b, k \in \mathbb{N}$:

$$p_b(X) = 1 + 2X + 2X^2 + \cdots + 2X^{b-1} + X^b = (1 - X^b)\frac{1+X}{1-X}, \tag{27}$$

$$q_b(X) = 1 + X^b, \tag{28}$$

$$r_{a,b}(X) = 1 + aX + aX^2 + \cdots + aX^{b-1} + X^b, \tag{29}$$

$$r_{a,b;\,k}(X) = 1 + aX + aX^2 + \cdots + aX^{b-1} + X^b + kX^{b/2} \quad (b \text{ even}). \tag{30}$$

Lemma C.6. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions, such that the following holds: f has t real zeroes $\rho_1 < \cdots < \rho_t$, and g has a transverse zero in each interval $]\rho_i, \rho_{i+1}[$ and no other zero in $[\rho_1, \rho_t]$. Then for any $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ the function $\alpha f + \beta g$ has at least $t-1$ real zeroes.*

Proof. The function $\alpha f + \beta g$ has opposite signs at ρ_i and ρ_{i+1}, so has a zero in $]\rho_i, \rho_{i+1}[$. □

Lemma C.7. *If $a \in \mathbb{Z}$, $b \in 2\mathbb{N}$, and $2 + a(b-1) < 0$, then $r_{a,b}$ is a reciprocal Salem polynomial.*

Proof. First write

$$r_{a,b}(X) = \frac{a}{2}p_b(X) + (1 - \frac{a}{2})q_b(X).$$

As p_b and q_b are symmetric polynomials, their Cayley transforms are polynomials with real coefficients. The zeroes of p_b are -1 and the $e^{2i\pi\frac{k}{b}}, k \in \{1, \dots, b-1\}$; the zeroes of q_b are the $e^{i\pi\frac{2k+1}{b}}, k \in \{0, \dots, b-1\}$. As these zeroes are interleaved on the unit circle, the zeroes of their Cayley transforms will be interleaved on the real axis, and we may apply

Lemma C.6 to conclude that $r_{a,b}(X)$ has at least $b-1$ zeroes on the unit circle. It has precisely $b-1$ zeroes there because $r_{a,b}(1) = 2 - a(b-1) < 0$ and $\lim_{x\to\infty} r_{a,b}(x) = +\infty$, so that $r_{a,b}$ has a zero in $]1, \infty[$. $\square$

LEMMA C.8. *Let $f(X) \in \mathbb{Z}[X]$ be a reciprocal Salem polynomial. Let $g(X) \in \mathbb{Z}[X]$ be a polynomial of degree less than $\deg(f)$, such that $f+g$ is reciprocal. Consider for $\epsilon \in \mathbb{R}$ the perturbation $f_\epsilon = f + \epsilon g \in \mathbb{R}[X]$. Let $k \in \mathbb{N}$ be such that f_ϵ has only simple roots for all $\epsilon \in [0, k]$. Then f_k is a reciprocal Salem polynomial.*

PROOF. Let F and F_ϵ be the Cayley transforms of f and f_ϵ. Then F has real roots except two which are complex conjugate, it is reciprocal, and has real coefficients. By assumption, the discriminant of F_ϵ does not change its sign on $[0, k]$ and thus F_k has real roots except two which are still complex conjugate. Taking the inverse Cayley transform yields f_k which has all its roots on the unit circle except two, and thus is a Salem polynomial. $\square$

LEMMA C.9. *If $a \in \mathbb{Z}$, $b \in 2\mathbb{N}$ with $2 + a(b-1) < 0$, and $k \in \mathbb{N}$ with $k \leq 2 - a$, then $r_{a,b;\,k}$ is a reciprocal Salem polynomial.*

PROOF. In view of Lemma C.8, it suffices to prove that $r_{a,b;\,\epsilon} = r_{a,b} + \epsilon X^{b/2}$ is simple for all $\epsilon \in [0, 2-a[$. For this purpose consider the function $f(X) = X^{\frac{b}{2}} r_{a,b}(X) = X^{-\frac{b}{2}}\left(1 + X + a\frac{1-X^b}{1-X}\right)$, considered as a function on the circle ($f : \mathbb{T} \to \mathbb{C}$).

Since f is real and $f(X^{-1}) = f(X)$, it satisfies $f(\mathbb{T}) \subset \mathbb{R}$: for any $\xi \in \mathbb{T}$,

$$\overline{f(\xi)} = \overline{f}(\overline{\xi}) = f(\xi^{-1}) = f(\xi).$$

We show that $r_{a,b;\,\epsilon}$ is simple by showing that $f + \epsilon$ has only simple zeroes on $\mathbb{T}$, or equivalently that f attains values greater than ϵ between its zeroes on $\mathbb{T}$.

Since $r_{a,b}$ is a reciprocal Salem polynomial, it has $b-2$ zeroes on $\mathbb{T}$, so f has also $b-2$ zeroes on $\mathbb{T}$. Consider the points $\xi_j = e^{2i\pi j/b} \in \mathbb{T}$, for $j \in \{1, \dots, b-1\}$. We have

$$f(\xi_j) = \xi_j^{-\frac{b}{2}}\left(\xi_j^b + 1 + a\frac{\xi_j^b - \xi_j}{\xi_j - 1}\right) = (-1)^j(2-a),$$

so the zeroes of f are separated by extrema of at least $\pm(2-a)$. The zeroes of $f+\epsilon$ (and thus of $r_{a,b;\,\epsilon}$) on $\mathbb{T}$ therefore remain simple. $\square$

We now turn to the factorization of the $r_{a,b}$. Let Q be any reciprocal Salem polynomial, let λ be the Salem number associated to Q, and let S be the minimal polynomial of λ. Then we have a factorization $Q = ST$, where T, having only roots on the unit circle, is a product of cyclotomic polynomials, in virtue of the theorem of Kronecker [**Kro99**, Vol. III, Part I, pages 47–110]. We show that this show that this factor T is either 1 or $X+1$, depending on the parity of b:

PROPOSITION C.10. *Suppose $|a-1| \geq 2$. Then the only cyclotomic polynomials dividing the $r_{a,b}$ defined in (29) are $X+1$ when b is odd, and $X^2 - X + 1$ when $a = -1$ and $b \equiv 2 \mod 6$, with the exception $a = -2, b = 2$ when $r_{a,b} = (X-1)^2$ is not a Salem polynomial.*

PROOF. Set $f(X) = r_{a,b}(X)$. Clearly $f(-1) = 0$ precisely when b is odd, and $f(1) \neq 0$. Suppose now that η is a root of unity of order $n > 2$ satisfying $f(\eta) = 0$. We may suppose, by direct computation, that n does not divide $b-1$. Let ξ be an algebraic conjugate of η satisfying $|\xi^{b-1} - 1| \geq 1$. Since

$$\left|\frac{\xi^{b+1}-1}{\xi^{b-1}-1}\right| = \left|\xi\left(1 - a + \frac{f(\xi)}{\xi + \cdots + \xi^{b-1}}\right)\right| = |a-1|,$$

and $|\xi^{b+1} - 1| \leq 2$, we have $|a-1| \leq 2$. Now in case $|a-1| = 2$ and ξ is such a root, we must have $|\xi^{b+1} - 1| = 2$ whence $\xi^{b+1} = -1$. Then we have $\xi^b - \xi = (1-a)/2 = \pm 1$, or, after multiplication by ξ and simplification, $\xi^2 \pm \xi + 1 = 0$, so ξ is of degree 3 or 6. We check by substitution in f that indeed $\xi^2 - \xi + 1$ divides f when $b \equiv 2 \mod 6$. We have shown that the only roots of unity of degree greater than 2 are the sixth, and that they occur only in very special cases. □

A similar, but more complicated, result holds for $r_{a,b;\,2}$; then there are special cases for $-3 \leq a \leq -1$, with factors T depending on the value of b modulo 4, 6 and 10, as mentioned in the statement of the theorem; we omit the uninteresting details and quote the result without proof.

Proof of Theorem C.1. If m is even the denominator of $F_{\ell,m}$ is $r_{2-\ell,\frac{m}{2}}$, by (25), and thus is a (reciprocal) Salem polynomial by Lemma C.7. If m is odd, this denominator is $r_{2-\ell,m-1;\,2}$, by (26), and thus is a (reciprocal) Salem polynomial, by Lemma C.9. It is irreducible, by Proposition C.10. □

C.4. The Growth of the Graphs $\mathcal{X}_{\ell,m}$

Recall the following well-known fact from [**GKP94**, p. 341]:

Lemma C.11. *Let $f(X) = P(X)/Q(X) = \sum_{n\geq 0} f_n X^n$ be a rational function of X, where P and Q are complex polynomials and $Q(X) = \prod_{i=1}^{r}(X-\alpha_i)^{\nu_i}$ with distinct $\alpha_i \in \mathbb{C}$, namely $\alpha_i \neq \alpha_j$ when $i \neq j$. Then there exist polynomials $R_1, \ldots, R_r \in \mathbb{C}[X]$ such that $f_n = \sum_{i=1}^{r} R_i(n)/\alpha_i^n$. Moreover the degree of R_i is strictly smaller than ν_i, for all i. In particular, if all poles of f are simple, then the R_i are constant.*

From this, we derive the following result:

Theorem C.12. *Let*

$$f(X) = \frac{P(X)}{Q(X)} = \sum_{n\geq 0} f_n X^n$$

be a rational function of X, where $P \in \mathbb{Z}[X]$ and Q is a Salem polynomial. Then there exist a constant $K > 0$ and a polynomial R such that for all n we have

$$K\lambda^n - R(n) < f_n < K\lambda^n + R(n),$$

where $\lambda > 1$ is the Salem number associated to Q. The degree of R is strictly less than the maximal multiplicity of f's poles. Thus if moreover all poles of f are simple, then there exist constants $\lambda > 1$, K and R such that

$$K\lambda^n - R < f_n < K\lambda^n + R.$$

Proof. Apply Lemma C.11 to f to obtain polynomials $R_1, \ldots, R_r$. Without loss of generality we may assume that α_r is the only pole of f outside the unit circle. Thus $K := R_r$ is a constant. Set also $\lambda = 1/\alpha_r$. Writing $R_i(n) = \sum b_{ij} n^j$, we define polynomials S_i by $S_i(n) = \sum |b_{ij}| n^j$, and we let

$$R(n) := \sum_{i=1}^{r-1} S_i(n).$$

Then $|R_i(n)/\alpha_i^n| \leq S_i(n)$, and

$$|f_n - K\lambda^n| = |f_n - \frac{R_r(n)}{\alpha_r^n}| \leq R(n).$$

If all poles of f are simple, then the R_i are constants and so is R. □

Corollary C.3 follows from the previous theorem.

ANNEXE D

Lie Methods in Growth of Groups and Groups of Finite Width

Laurent Bartholdi and Rostislav I. Grigorchuk

D.1. Introduction

The main goal of this paper is to present new examples of groups of finite width and to give a method of proving that some groups from the class of branch groups have finite width. This provides examples of groups of finite width with a completely new origin and answers a question asked by several mathematicians. We also give new examples of Lie algebras of finite width associated to the groups mentioned above.

The first group we study, $\mathfrak{G}$, was constructed in [**Gri80a**] where it was shown to be an infinite torsion group; later in [**Gri84**] it was shown to be of intermediate growth. The second group, $\widetilde{\mathfrak{G}}$, was already considered by the second author in 1979, but was rejected at that time for not being periodic. We now know that it also has intermediate growth [**BG99a**] and finite width.

Our interest in the finite width property comes from the theory of growth of groups. Another important area connected to this property is the theory of finite p-groups and the theory of pro-p-groups; see [**Sha95b**], [**Sha95a**, §8] and [**KLP97**] with its bibliography. More precisely, the following was discussed by many mathematicians and stated by Zel'manov in Castelvecchio in 1996 [**Zel96**]:

Conjecture D.1. *Let G be a just-infinite pro-p-group of finite width. Then G is either solvable, p-adic analytic, or commensurable to a positive part of a loop group or to the Nottingham group.*

Our computations disprove this conjecture by providing a counter-example, the profinite completion of $\mathfrak{G}$ (it is a pro-p-group with $p = 2$). Note that it exhibits a behaviour specific to positive characteristic: indeed it was proved by Martinez and Zel'manov in [**MZ99**] that unipotence and finite width imply local nilpotence.

Before we give the definition of a group of finite width, let us recall a classical construction of Magnus [**Mag40**], described for instance in [**HB82**, Chapter VIII]. Given a group G and $\{G_n\}_{n=1}^{\infty}$ an N-series (i.e. a series of normal subgroups with $G_1 = G$, $G_{n+1} \leq G_n$ and $[G_m, G_n] \leq G_{m+n}$ for all $m, n \geq 1$), there is a canonical way of associating to G a graded Lie ring

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} L_n, \tag{31}$$

where $L_n = G_n/G_{n+1}$ and the bracket operation is induced by commutation in G. Possible examples of N-series are the lower central series $\{\gamma_n(G)\}_{n=1}^{\infty}$; for an integer p, the *lower p-central series* given by $P_1(G) = G$ and $P_{n+1}(G) = P_n(G)^p[P_n(G), G]$; and, for a field $\Bbbk$, the series of $\Bbbk$*-dimension subgroups* $\{G_n\}_{n=1}^{\infty}$ defined by

$$G_n = \{g \in G |\, g - 1 \in \Delta^n\}, \qquad n = 1, 2, \dots$$

where Δ is the augmentation (or fundamental) ideal of the group algebra $\Bbbk[G]$.

Tensoring the $\mathbb{Z}$-modules L_n with a suitable field $\Bbbk$, we obtain in (31) a graded Lie algebra $\mathcal{L}_{\Bbbk}(G)$. In case the N-series chosen satisfies the additional condition $G_n^p \leq G_{pn}$, and $\Bbbk$ is a field of characteristic p, the algebra $\mathcal{L}_{\Bbbk}(G)$ will then be a p-algebra or *restricted algebra*; see [**Jac41**] or [**Jac62**, Chapter V], the Frobenius operation on $\mathcal{L}_{\Bbbk}(G)$ being induced by raising to the power p in G. In this case the quotients G_n/G_{n+1} are elementary p-groups.

Many properties of a group are reflected in properties of its corresponding Lie algebra. For instance, one of the most important results obtained using the Lie method is the theorem of Zel'manov [**Zel95a**] asserting that if the Lie algebra $\mathcal{L}_{\mathbb{F}_p}(G)$ associated to the dimension subgroups of a finitely generated periodic residually-p group G satisfies a polynomial identity then the group G is finite ($\mathbb{F}_p$ is the prime field of characteristic p). This result gives in fact a positive solution to the Restricted Burnside Problem [**VZ93, Zel95b, VZ96, Zel97**]. Another example is the criterion of analyticity of pro-p-groups discovered by Lazard [**Laz65**].

The Lie method also applies to the theory of growth of groups, as was first observed in [**Gri89**]. There the second author proved that in the class of residually-p groups there is a gap between polynomial growth and growth of type $e^{\sqrt{n}}$. This result was then generalized in [**LM91**, Theorem D] to the class of residually-nilpotent groups, and in [**CG97**] the Lie method was also used to prove that certain one-relator groups with exponential-growth Lie algebra $\mathcal{L}_{\Bbbk}(G)$ have uniformly exponential growth. If a group G is finitely generated, then its Lie algebra $\mathcal{L}_{\Bbbk}(G) = \bigoplus L_n \otimes \Bbbk$ is also finitely generated, and the growth of $\mathcal{L}_{\Bbbk}(G)$ is by definition the growth of the sequence $\{b_n = \dim(L_n \otimes \Bbbk)\}_{n=1}^{\infty}$.

The investigation of the growth of graded algebras related to groups has its own interest and is related to other topics. One of the first results in this direction is the Golod-Shafarevich inequality [**GS64**] which plays an important role in group, number and field theories. The idea of Golod and Shafarevich was used by Lazard in the proof of the aforementioned criterion of analyticity (he even used the notation 'gosha' for the growth of the algebras). Vershik and Kaimanovich observed the relation between the growth of gosha, amenability, and asymptotic behaviour of random walks (see Section D.4 below).

For our purposes it will be sufficient to consider only the fields $\mathbb{Q}$ and $\mathbb{F}_p$. Let G_n be the corresponding series of dimension subgroups, which is also an N-series, and let $\mathcal{L}_{\Bbbk}(G)$ be the associated Lie algebra. If $\mathcal{L}_{\Bbbk}(G)$ is of polynomial growth of degree $d \geq 0$, then the growth of G is at least $e^{n^{1-1/(d+2)}}$, and if $\mathcal{L}_{\Bbbk}(G)$ is of exponential growth, then G is of uniformly exponential growth.

If $\Bbbk = \mathbb{Q}$ and G is residually-nilpotent and $b_n = 0$ for some n, then G is nilpotent; indeed G_n must be finite for that n, whence $\gamma_n(G)$ is finite too, and since $\bigcap_{k\geq 1} \gamma_k(G) = 1$ this implies that $\gamma_N(G) = 1$ for some N. It follows that G has polynomial growth [**Mil68a**]. In fact polynomial growth is equivalent to virtual nilpotence [**Gro81a**].

If $\Bbbk = \mathbb{F}_p$ and G is a residually-p group and $b_n = 0$ for some n, then G is a linear group over a field, by Lazard's theorem [**Laz65**] and therefore has either polynomial or exponential growth, by the Tits alternative [**Tit72**].

Finally, if $b_n \geq 1$ for all n then, independent of $\Bbbk$, the growth of G is at least $e^{\sqrt{n}}$. Keeping in mind that polynomial growth $b_n \sim n^d$ of $\mathcal{L}_{\Bbbk}(G)$ implies a lower bound $e^{n^{1-1/(d+2)}}$ for the growth of G, we conclude that examples of groups with growth exactly $e^{\sqrt{n}}$ are to be found amongst the class of groups for which the sequence $\{b_n\}_{n=1}^{\infty}$ is uniformly bounded, or at least bounded in average. This key observation leads to the notion of *groups of finite width.* We present two versions of the definition:

DEFINITION D.2. *Let G be a group and $\Bbbk \in \{\mathbb{Q}, \mathbb{F}_p\}$ a field. If $\Bbbk = \mathbb{Q}$, assume G is residually-nilpotent; if $\Bbbk = \mathbb{F}_p$, assume G is residually-p.*

(1) *G has* finite C-width *if there is a constant K with $[\gamma_n(G) : \gamma_{n+1}(G)] \leq K$ for all n.*

(2) *G has* finite D-width *with respect to $\Bbbk$ if there is a constant K with $b_n \leq K$ for all n, where $\{b_n\}_{n=1}^{\infty}$ is the growth of $\mathcal{L}_{\Bbbk}(G)$ constructed from the dimension subgroups.*

A third notion can be defined, that of *finite averaged width*; see [**Gri89**] or [**KLP97**, Definition I.1.ii]. From our point of view D-width is more natural; but the first notion is more commonly used in the theory of finite p-groups and pro-p-groups, see for instance [**KLP97**, Definition I.1.i]. The examples we will produce are of finite width according to both definitions. That one of our groups has finite width was conjectured in [**Gri89**]; it was proven that the numbers b_n are bounded in average. Rozhkov then confirmed this conjecture in [**Roz96a**] by computing explicitly the b_n; but the proof had gaps, one of which was filled in [**Roz96b**]. We fix another gap in the "Technical Lemma 4.3.2" of [**Roz96b**] while simplifying and clarifying Rozhkov's proof, and also outline a general method, connected to ideas of Kaloujnine [**Kal46**].

We recall in the next section known notions on algebras associated to groups, and construct in Section D.3 a torsion group of uniformly exponential growth. Section D.5 describes a class of groups acting on rooted trees, and the next two sections detail for two specific examples the indices of the lower central and dimensional series. More specifically, we compute in Theorem D.23 and D.30 the indices of these series for the group $\mathfrak{G}$ and an overgroup $\widetilde{\mathfrak{G}}$. We also obtain in the process the structure of the Lie algebras $L(\mathfrak{G})$ (associated to the lower central series) and $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$ (associated to the dimension series) in Theorem D.24, and that of $L(\widetilde{\mathfrak{G}})$ and $\mathcal{L}_{\mathbb{F}_2}(\widetilde{\mathfrak{G}})$ in Theorem D.31. They are described using Cayley graphs of Lie algebras, see Subsection D.6.1.

Throughout this paper groups shall act on the left. We use the notational conventions $[x, y] = xyx^{-1}y^{-1}$ and $x^y = yxy^{-1}$.

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D.2. Growth of Groups and Associated Graded Algebras

Let G be a group, $\{\gamma_n(G)\}_{n=1}^{\infty}$ the lower central series of G, $\Bbbk \in \{\mathbb{Q}, \mathbb{F}_p\}$ a prime field, $\Delta = \ker(\varepsilon) < \Bbbk[G]$ the augmentation ideal, where $\varepsilon(\sum k_i g_i) = \sum k_i$ is the augmentation map $\Bbbk[G] \to \Bbbk$, and $\{G_n\}_{n=1}^{\infty}$ the series of dimension subgroups of G [**Zas40, Jen41**]. Recall that

$$G_n = \{g \in G \mid g - 1 \in \Delta^n\}.$$

The restrictions we impose on $\Bbbk$ are not important, as G_n depends only on the characteristic of $\Bbbk$. We suppose throughout that G is residually-nilpotent if $\Bbbk = \mathbb{Q}$ and is residually-p if $\Bbbk = \mathbb{F}_p$.

If $\Bbbk = \mathbb{Q}$, then G_n is the isolator of $\gamma_n(G)$, as was proved in [**Jen55**] (see also [**Pas77**, Theorem 11.1.10] or [**Pas79**, Theorem IV.1.5]); i.e.

$$G_n = \sqrt{\gamma_n(G)} = \{g \in G \mid g^\ell \in \gamma_n(G) \text{ for an } \ell \in \mathbb{N}\}.$$

Note that in [**Pas77**] these results are stated for finite p-groups. They nevertheless hold in the more general setting of residually-nilpotent or residually-p groups.

If $\Bbbk = \mathbb{F}_p$, then $\gamma_n(G) \leq G_n \leq \sqrt{\gamma_n(G)}$, and the G_n can be defined in several different ways, for instance by the relation

$$G_n = \prod_{i \cdot p^j \geq n} \gamma_i^{p^j}(G)$$

due to Lazard [**Laz53**], or recursively as

$$G_n = [G, G_{n-1}] G^p_{\lceil n/p \rceil}, \tag{32}$$

where $\lceil n/p \rceil$ is the least integer greater than or equal to n/p. In characteristic p, the series $\{G_n\}_{n=1}^{\infty}$ is called the lower p-central, Brauer, Jennings, Lazard or Zassenhaus series of G. The quotients G_n/G_{n+1} are elementary abelian p-groups and define the fastest-decreasing central series with the property $G_n^p \le G_{np}$ [**Jen55**].

Let

$$\mathcal{A}(G) = \mathcal{A}_{\Bbbk}(G) = \bigoplus_{n=0}^{\infty} \Delta^n/\Delta^{n+1}$$

be the associative graded algebra with product induced linearly from the group product (see [**Pas77, Pas79**] for more details).

If $\Bbbk = \mathbb{Q}$, consider the following graded Lie algebras over $\Bbbk$:

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} \left(G_n/G_{n+1} \otimes_{\mathbb{Z}} \mathbb{Q}\right), \qquad L(G) = \bigoplus_{n=1}^{\infty} \left(\gamma_n(G)/\gamma_{n+1}(G) \otimes_{\mathbb{Z}} \mathbb{Q}\right).$$

If $\Bbbk = \mathbb{F}_p$, consider the restricted Lie $\mathbb{F}_p$-algebra

$$\mathcal{L}_p(G) = \bigoplus_{n=1}^{\infty} \left(G_n/G_{n+1}\right).$$

Then Quillen's Theorem [**Qui68**] asserts that $\mathcal{A}(G)$ is the universal enveloping algebra of $\mathcal{L}(G)$ in characteristic 0 and is the universal p-enveloping algebra of $\mathcal{L}_p(G)$ in positive characteristic.

Let us introduce the following numbers:

$$a_n(G) = \dim_{\Bbbk}(\Delta^n/\Delta^{n+1}), \qquad b_n(G) = \operatorname{rank}(G_n/G_{n+1}).$$

Here by the rank of the G-module M we mean the torsion-free rank $\dim_{\mathbb{Q}}(M \otimes \mathbb{Q})$ in characteristic 0 and the p-group rank $\dim_{\mathbb{F}_p}(M \otimes \mathbb{F}_p)$, equal to the minimal number of generators, in positive characteristic. Note that in zero-characteristic $b_n = \operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))$, because the natural map

$$\gamma_n(G)/\gamma_{n+1}(G) \to G_n/G_{n+1}$$

has finite kernel and cokernel.

The following result is due to Jennings. The case $\Bbbk = \mathbb{F}_p$ appears in [**Jen41**] and the case $\Bbbk = \mathbb{Q}$ appears in [**Jen55**]; but see also [**Pas77**, Theorem 3.3.6 and 3.4.10].

$$\sum_{n=0}^{\infty} a_n(G)t^n = \begin{cases} \prod_{n=1}^{\infty} \left(\frac{1-t^{pn}}{1-t^n}\right)^{b_n(G)} & \text{if } \Bbbk = \mathbb{F}_p, \\ \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n(G)} & \text{if } \Bbbk = \mathbb{Q}. \end{cases} \tag{33}$$

The series $\sum_{n=0}^{\infty} a_n(G)t^n$ is the Hilbert-Poincaré series of the graded algebra $\mathcal{A}(G)$. The equation (33) expresses this series in terms of the numbers $b_n(G)$; the relation between the sequences $\{a_n(G)\}_{n=0}^{\infty}$ and $\{b_n(G)\}_{n=1}^{\infty}$ is quite complicated. We shall be interested in asymptotic growth of series, in the following sense:

DEFINITION D.3. *Let f and g be two functions $\mathbb{R}_+ \to \mathbb{R}_+$. We write $f \precsim g$ if there is a constant $C > 0$ such that $f(x) \le C + Cg(Cx + C)$ for all $x \in \mathbb{R}_+$, and write $f \sim g$ if $f \precsim g$ and $g \precsim f$.*

A series $\{a_n\}_{n=0}^{\infty}$ defines a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(x) = a_{\lfloor x \rfloor}$, and for two series $a = \{a_n\}$ and $b = \{b_n\}$ we write $a \precsim b$ and $a \sim b$ when the same relations hold for their associated functions.

The main facts are presented in the following statement:

PROPOSITION D.4. *Let $\{a_n\}$ and $\{b_n\}$ be connected by the one of the relations (33). Then*

(1) *$\{b_n\}$ grows exponentially if and only if $\{a_n\}$ does, and we have*

$$\limsup_{n\to\infty} \frac{\ln a_n}{n} = \limsup_{n\to\infty} \frac{\ln b_n}{n}.$$

(2) *If $b_n \sim n^d$ then $a_n \sim e^{n^{(d+1)/(d+2)}}$.*

PROOF. We first suppose $\Bbbk = \mathbb{Q}$, and prove Part 1 following [**Ber83**]. Let $A = \limsup(\ln a_n)/n$ and $B = \limsup(\ln b_n)/n$. Clearly $A \geq B$ as $a_n \geq b_n$ for all n; we now prove that $A \leq B$. Define

$$f(z) = \prod_{n=1}^{\infty}(1 - e^{-nz})^{-b_n},$$

viewed as a complex analytic function in the half-plane $\Re(z) > B$. We have $|1-e^{-nz}|^{-1} \leq (1 - e^{-n\Re z})^{-1}$, from which $|f(z)| \leq f(\Re z)$. Now applying the Cauchy residue formula,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u+iv)e^{n(u+iv)}dv \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(u+iv)|e^{nu}dv \leq e^{nu}f(u)$$

for all $u > B$, so

$$A = \limsup_{n\to\infty}\frac{\ln a_n}{n} \leq \limsup_{u>B,n\to\infty}\left(u + \frac{\ln f(u)}{n}\right) = B.$$

For $\Bbbk = \mathbb{F}_p$, Part 1 holds *a fortiori*.

Part 2 for $\Bbbk = \mathbb{Q}$ is a consequence of a result by Meinardus ([**Mei54**]; see also [**And76**, Theorem 6.2]). More precisely, when $b_n = n^d$, his result implies that

$$a_n \approx \frac{e^{\zeta'(-d)}}{\sqrt{2\pi(d+2)n}}\left(\frac{(d+1)!\zeta(d+2)}{n}\right)^{\frac{1-2\zeta(-d)}{2+4d}} e^{n\frac{d+2}{d+1}\left(\frac{(d+1)!\zeta(d+2)}{n}\right)^{\frac{1}{1+2d}}},$$

where '$\approx$' means that the quotient tends to 1 as $n \to \infty$, and ζ is the Riemann zeta function.

We sketch the proof for $\Bbbk = \mathbb{Q}$ below: we suppose that $b_n \sim n^d$, so $A = B = 0$ by Part 1, and compute

$$\begin{aligned}\frac{d}{du}\ln f(u) = \sum_{n=1}^{\infty} -b_n\frac{-ne^{-nu}}{1-e^{-nu}} \quad &\sim \quad \frac{1}{u^{d+2}}\sum_{n=1}^{\infty}\frac{(nu)^{d+1}}{e^{nu}-1}u \\ &\sim \quad \frac{1}{u^{d+2}}\int_0^{\infty}\frac{w^{d+1}}{e^w-1}dw = \frac{C}{u^{d+2}}.\end{aligned}$$

Thus $\ln f(u) \sim C/u^{d+1}$, and the inequality

$$\log a_n \leq nu + \log f(u) \sim nu + C/u^{d+1}$$

is tight by the saddle-point principle when the right-hand side is minimized. This is done by choosing $u = n^{-1/(d+2)}$, whence as claimed $\log a_n \sim n^{1-1/(d+2)}$.

Finally, we show that (33) yields the same asymptotics when $\Bbbk = \mathbb{F}_p$ as when $\Bbbk = \mathbb{Q}$. Clearly

$$\prod_{n=1}^{\infty}(1+t^n)^{b_n} \leq \prod_{n=1}^{\infty}(1+t^n+\cdots+t^{(p-1)n})^{b_n} \leq \prod_{n=1}^{\infty}(1+t^n+\dots)^{b_n}$$

for all $p \geq 2$, where for two power series $\sum e_t^n$ and $\sum f_n t^n$ the inequality $\sum e_t^n \leq \sum f_n t^n$ means that $e_n \leq f_n$ for all n. It thus suffices to consider the case $p = 2$. For this purpose define

$$g(z) = \prod_{n=1}^{\infty}(1 + e^{-nz})^{b_n},$$

and compare the series developments of $\log(f)$ and $\log(g)$ in e^{-z}. From $-\log(1-z) = \sum_{n\geq 1}\frac{z^n}{n}$ it follows that

$$\log f(z) = \sum_{n\geq 1} f_n e^{-nz}, \quad f_n = \sum_{d|n}\frac{1}{d},$$

$$\log g(z) = \sum_{n\geq 1} g_n e^{-nz}, \quad g_n = \sum_{d|n}\frac{(-1)^{d+1}}{d},$$

so both series have the same odd-degree coefficients, and thus $\log f \sim \log g$. Their exponentials then have the same asymptotics; more precisely, $f_n \leq g_{2n-1}$ for all n, so $e^z \log f(2z) \leq \log g(z)$ termwise, and $f(2z) \leq g(z)$. ☐

D.2.1. Growth of Groups. Let G be a finitely generated group with a fixed semigroup system S of generators (i.e. such that every element $g \in G$ can be expressed a product $g = s_1 \dots s_n$ for some $s_i \in S$). Let $\gamma_G^S(n)$ be the growth function of (G, S); recall that it is

$$\gamma_G^S(n) = \#\{g \in G \mid |g| \leq n\},$$

where $|g|$ is the minimal number of generators required to express g as a product.

The following observations are well-known:

LEMMA D.5. *Let G be a group and consider two finite generating sets S and T. Then $\gamma_G^S \sim \gamma_G^T$, with $\sim$ given in Definition D.3.*

It is then meaningful to consider the *growth* γ_G of G, which is the $\sim$-equivalence class containing its growth functions γ_G^S.

LEMMA D.6. *Let G be a finitely generated group, $H < G$ a finitely generated subgroup and K a quotient of G. Then $\gamma_H \precsim \gamma_G$ and $\gamma_K \precsim \gamma_G$.*

PROOF. Let S be a finite generating set for H; choose a generating set $T \supset S$ for G. Apply Definition D.3 with $C = 1$ to obtain $\gamma_H^S \precsim \gamma_G^T$. Clearly $\gamma_K^T(n) \leq \gamma_G^T(n)$ for all n. ☐

LEMMA D.7 ([**Gri89**]). *For any field $\mathbb{k}$ and any group G with generating set S the inequalities $a_n(G) \leq \gamma_G^S(n)$ hold for all $n \geq 0$.*

PROOF. Fix a generating set S. The identities

$$xy - 1 = (x-1) + (y-1) + (x-1)(y-1), \qquad x^{-1} - 1 = -(x-1) - (x-1)(x^{-1}-1)$$

show that

$$xy - 1 \equiv (x-1) + (y-1), \qquad x^{-1} - 1 \equiv -(x-1) \mod \Delta^2,$$

so Δ^n is generated over $\mathbb{k}$ by Δ^{n+1} and elements of the form

$$x_0(s_1-1)x_1(s_2-1)\dots(s_n-1)x_n,$$

for all $s_i \in S$ and $x_i \in \mathbb{k}[G]$. Now $x_i \equiv \varepsilon(x_i) \in \mathbb{k}$ modulo Δ, so Δ^n/Δ^{n+1} is spanned by the

$$(s_1-1)(s_2-1)\dots(s_n-1), \qquad s_i \in S.$$

All these elements are in the vector subspace S_n of $\mathbb{k}[G]$ spanned by products of at most n generators, and by definition S_n is of dimension $\gamma_G^S(n)$. ☐

COROLLARY D.8. $\{a_n(G)\}_{n=0}^\infty \precsim \gamma_G$.

Combining Proposition D.4 and Lemma D.7, we obtain as

COROLLARY D.9. *If there exist $C > 0$ and $d \geq 0$ such that $b_n \geq Cn^d$ for all n, then $\gamma_G(n) \succsim e^{1-1/(d+2)}$. In particular, if $b_n \neq 0$ for all n, then $\gamma_G(n) \succsim e^{\sqrt{n}}$.*

We shall say a group G is of *subradical growth* if $\gamma_G \not\succsim e^{\sqrt{n}}$.

THEOREM D.10 ([**Gri89**]). *Let G be a finitely generated residually-p group. If G is of subradical growth then G is virtually nilpotent and $\gamma_G(n) \sim n^d$ for some $d \in \mathbb{N}$.*

PROOF. By the previous corollary, $b_n(G) = 0$ for some n. Consider the p-completion $\widehat{G}$ of G. As Lie algebras, $\mathcal{L}_{\mathbb{F}_p}(G)$ and $\mathcal{L}_{\mathbb{F}_p}(\widehat{G})$ coincide, so $b_n(\widehat{G}) = 0$. By Lazard's criterion $\widehat{G}$ is an analytic pro-p-group [**Laz65**] and thus is linear over a field. Since G is residually-p it embeds in $\widehat{G}$ so is also linear. By the Tits alternative [**Tit72**] either G contains a free group on two generators (contradicting the assumption on the growth of G) or G is virtually solvable. By the results of Milnor and Wolf every virtually solvable group is either of exponential growth or is virtually nilpotent [**Mil68a, Wol68**]. The asymptotic growth is invariant under taking finite-index subgroups, and the growth of a nilpotent group is polynomial of degree $\sum_{k\geq 1} kb_k$, as was shown by Guivarc'h and Bass [**Gui70, Gui73, Bas72**]. □

In the class of residually-p groups, Theorem D.10 improves Gromov's result [**Gro81a**] that a finitely generated group G having polynomial growth is virtually nilpotent, in that the assumption is weakened from 'polynomial growth' to 'subradical growth'. Lubotzky and Mann have shown the same result for residually nilpotent groups of subradical growth. It is not known whether subradical growth does imply virtual nilpotence, and whether there exist groups of precisely radical growth. Certainly the right place to look for such examples is among groups of finite width, or groups satisfying some tight condition on the growth of their b_n.

Therefore new examples of groups of finite width are of special interest. Below we shall give two examples of such groups and outline a method of constructing new examples; but first a consequence of D.10 is

THEOREM D.11. *The growth $\gamma_{\mathfrak{G}}$ of the group $\mathfrak{G}$ satisfies*

$$e^{\sqrt{n}} \precsim \gamma_{\mathfrak{G}}(n) \precsim e^{n^{1/(1-\log_2 \eta)}},$$

where η is the real root of X^3+X^2+X-2.

PROOF. If $\mathfrak{G}$ were nilpotent it would be finite, as it is finitely generated and torsion; since it is infinite D.10 yields the left inequality.

The right inequality was proven by the first author in [**Bar98**], using purely combinatorial techniques. □

Note that the estimate from below can be obtained directly as in [**Gri84**], by showing that for an appropriate S the growth function γ_G^S satisfies

$$\gamma_G^S(4n) \geq \gamma_G^S(n)^2.$$

The second author conjectured in 1984 that the left inequality is in fact an equality, but Leonov recently announced that this is not the case [**Leo98b**].

For our second example $\widetilde{\mathfrak{G}}$ it is only known that

$$e^{\sqrt{n}} \precsim \gamma_{\widetilde{\mathfrak{G}}} \not\precsim e^n,$$

as is shown in [**BG99a**].

Lemma D.7 can also be used to study uniformly exponential growth, as was observed in [**CG97**]. Let

$$\omega_G^S = \lim_{n\to\infty} \sqrt[n]{\gamma_G^S(n)}$$

be the base of exponential growth of G with respect to the generating set S and let $\omega_G = \inf_S \omega_G^S$, the infimum being taken over all finite generating sets.

DEFINITION D.12. *The group G has* uniformly exponential growth *if $\omega_G > 1$.*

(See [**Gro81b**] for the original definition and motivations, and [**GH97**] for more details on this notion.) For instance, the free groups of rank ≥ 2, and more generally, the non-elementary hyperbolic groups have uniformly exponential growth [**Kou98**]. It is currently not known whether there exists a group of exponential but not uniformly exponential growth.

COROLLARY D.13. *If for some $\Bbbk \in \{\mathbb{Q}, \mathbb{F}_p\}$ the algebra $\mathcal{A}_{\Bbbk}(G)$ has exponential growth then G has uniformly exponential growth. (We do not need here the assumption that G is residually-p or residually nilpotent.)*

In the next section we will combine this idea with the Golod-Shafarevich construction to produce examples of finitely generated residually finite p-groups of uniformly exponential growth.

D.3. Torsion Groups of Uniformly Exponential Growth

As a reference to the Golod-Shafarevich construction we recommend the original paper [**GS64**], one of the books [**Her94, Koc70**], or [**HB82**, § VIII.12].

Consider the free associative algebra A over the field $\mathbb{F}_p$ on the generators $x_1, \dots, x_d$ for some $d \geq 2$. The algebra A is graded: $A = \bigoplus_{n=0}^{\infty} A_n$ where A_n is spanned by the monomials of degree n, with $A_0 = \mathbb{F}_p 1$. Elements of the subspace A_n are called *homogeneous of degree n*.

Consider an ideal $\mathcal{I}$ in A generated by r_1 homogeneous elements of degree 1, r_2 of degree 2, etc. (We make this homogeneity assumption for simplicity; it is not necessary, as was indicated in [**Koc70**].) Let $B = A/\mathcal{I}$. Then B is also a graded algebra: $B = \bigoplus_{n=0}^{\infty} B_n$ and if $H_B(t) = \sum_{n=0}^{\infty} d_n t^n$ be the Hilbert-Poincaré series of B, i.e. $d_n = \dim_{\mathbb{F}_p} B_n$, then the Golod-Shafarevich inequality

$$H_B(t)(1 - dt + H_R(t)) \geq 1 \tag{34}$$

holds; here $H_R(t) = \sum_{n=1}^{\infty} r_n t^n$, and for the comparison of two power series the same agreement holds as in the previous section.

Suppose that for some $\xi \in (0,1)$ the series $H_R(t)$ converges at ξ and $1 - d\xi + H_R(\xi) \leq 0$. Then the series $H_B(t)$ cannot converge at $t = \xi$, so the coefficients d_n of $H_B(t)$ grow exponentially and

$$\limsup_{n\to\infty} \sqrt[n]{d_n} \geq \frac{1}{\xi}.$$

Golod proves in [**Gol64**] that $\mathcal{I}$ can be chosen in such a way that the ideal $\mathcal{D} = \bigoplus_{n=1}^{\infty} B_n$ will be a p-nilalgebra (i.e. for all $y \in \mathcal{D}$ there is an $n \in \mathbb{N}$ such that $y^{p^n} = 0$).

The construction of the relators goes as follows: enumerate first as $\{y_k\}_{k=1}^{\infty}$ all elements of the algebra A (this is possible since A is countable). Start with $\mathcal{I}_0 = 0$; then if y_k is not a nilelement of $A/\mathcal{I}_{k-1}$ take $\ell_k \geq 3$ sufficiently large so that the least degree of monomials in $y_k^{p^{\ell_k}}$ is larger than all degrees of monomials in $\mathcal{I}_{k-1}$. Construct $\mathcal{I}_k$ by adding to $\mathcal{I}_{k-1}$ all homogeneous parts of the polynomial $y_k^{p^{\ell_k}}$. Let finally $\mathcal{I} = \bigcup_{n=0}^{\infty} \mathcal{I}_n$.

The numbers r_k will then all be 0 or 1 with $r_k = 0$ for $k < p^3$, so taking $\xi = 3/4$ we have

$$1 - d\xi + H_R(\xi) \leq 1 - 2\xi + \frac{\xi^{2^3}}{1-\xi} < 0$$

and $B = A/\mathcal{I}$ is of exponential growth at least $(4/3)^n$. Let $\overline{x}_1, \dots, \overline{x}_d$ be the images of $x_1, \dots, x_d$ in B, and let G be the group generated by the elements $s_i = 1 + \overline{x}_i$; they are invertible because the $\overline{x}_i$ are p-nilelements and B is of characteristic p. The vector subspace of B spanned by G is B itself, so B is a quotient of the group algebra $\mathbb{F}_p[G]$.

THEOREM D.14. *G is a finitely generated residually finite p-group of uniformly exponential growth.*

PROOF. That G is a p-group was observed by Golod and follows from the fact that $\mathcal{D} = \bigoplus_{n=1}^{\infty} B_n$ is a p-nilalgebra. Let π be the natural map $\mathbb{F}_p[G] \to B$. Then $\mathcal{D}$ is generated by $\pi(\Delta)$ and more generally $\bigoplus_{n=N}^{\infty} B_n = \pi(\Delta^N)$, so by Lemmata D.7 and D.6 there is a $\xi < 1$ such that the estimate

$$\frac{1}{\xi^n} \leq \dim_{\mathbb{F}_p} B_n \leq a_n(G) \leq \gamma_G^T(n), \qquad n = 1, 2, \dots$$

holds for any system T of generators of G. □

D.4. Growth of Algebras and Amenability

As was mentioned in the introduction, there is an interesting question (due to Vershik) on the relation between the amenability of a group and the growth of related algebras. Let us formulate our version of this question:

PROBLEM D.15. 1. *Let G be amenable. Does $b_n(G)$ grow subexponentially for any field $\Bbbk$?*
2. *Suppose G is residually nilpotent (or residually-p) and $b_n(G)$ grows subexponentially for the field $\mathbb{Q}$ (or $\mathbb{F}_p$). Is then the group G amenable?*

There is a chance that for at least one of these questions the answer is affirmative.

For solvable groups (which are amenable) the associated algebras have subexponential growth, as follows from computations by Petrogradskiĭ [**Pet93, Pet96**]; his results are based on computations for free polynilpotent algebras by Bokut' [**Bok63**]. See also Egorychev [**Ego84**] and Berezniĭ [**Ber83**] for partial results.

On the other hand there is some similarity between the asymptotics of random walks on solvable groups and the growth of $b_n(G)$ [**Kaĭ80**] which gives a hope that subexponential growth of algebras implies (under the residuality hypothesis) subexponential decay of the probability of returning to the origin for symmetric random walks on a group. Then Kesten's criterion [**Kes59**] can be invoked to imply the amenability of G.

D.5. Groups Acting on Rooted Trees

We now consider examples of groups whose lower central series and dimension series we can compute explicitly. Let Σ be a finite alphabet, and Σ^* the set of finite sequences over Σ. This set has a natural rooted tree structure: the vertices are finite sequences, and the edges are all the $(\sigma, \sigma s)$ for $\sigma \in \Sigma^*$ and $s \in \Sigma$; the root vertex is $\emptyset$, the empty sequence. By $\mathsf{Aut}(\Sigma^*)$ we mean the bijections of Σ^* that preserve the tree structure, i.e. preserve length and prefixes. We write $\sigma\Sigma^*$ for the subtree of Σ^* below vertex σ: it is isomorphic to Σ^* but rooted at σ.

Let G be a finitely generated subgroup of $\mathsf{Aut}(\Sigma^*)$ acting transitively on Σ^n for all n (such an action will be called *spherically transitive*.) We denote by $\mathsf{Stab}_G(\sigma)$ the stabilizer of the vertex σ in G, and by $\mathsf{Stab}_G(n)$ the stabilizer of all vertices of length n. An arbitrary element $g \in \mathsf{Stab}_G(n)$ can be identified with a tuple $(g_\sigma)_{|\sigma|=n}$ of tree automorphisms; we write this monomorphism

$$\phi_n : \mathsf{Stab}_G(n) \hookrightarrow \prod_{\sigma\in\Sigma^n} \mathsf{Aut}(\sigma\Sigma^*).$$

We define the *vertex group* or *rigid stabilizer* $\mathsf{Rist}_G(\sigma)$ of the vertex σ by

$$\mathsf{Rist}_G(\sigma) = \{g \in G | \, g\tau = \tau \ \forall \tau \in \Sigma^* \setminus \sigma\Sigma^*\},$$

and the n^{th} *rigid stabilizer* as the group generated by the length-n vertex groups: $\mathsf{Rist}_G(n) = \langle \mathsf{Rist}_G(\sigma) : |\sigma| = n\rangle$. Since G acts transitively on Σ^n the vertex groups of vertices at level n are all conjugate. Therefore $\mathsf{Rist}_G(n)$ is a direct product of $|\Sigma|^n$ copies of $\mathsf{Rist}_G(\sigma)$ for a σ of length n.

DEFINITION D.16. *A finitely generated group G is called a* branch group *if*

(1) *G acts faithfully on Σ^* and transitively on Σ^n for all $n \geq 0$;*
(2) *$[G : \mathsf{Rist}_G(n)]$ is finite for all $n \geq 0$.*

D.5.1. The Modules V_n.

Let G be a group acting on a regular rooted tree Σ^*, where Σ contains p elements for some prime p; for ease of notation suppose $\Sigma = \mathbb{F}_p$. Assume moreover that at each vertex G acts as a power of the cyclic permutation $\epsilon = (0, 1, \dots, p-1)$ of Σ. Let $V_n = \mathbb{F}_p[G/\mathsf{Stab}_G(0^n)]$; it is a vector space of dimension p^n, as G acts transitively on Σ^n, and has a natural G-module structure coming from the action of G on $G/\mathsf{Stab}_G(0^n)$. Identify $G/\mathsf{Stab}_G(0^n)$ with the set Σ^n of vertices at level n, and also with the set of monomials over $\{X_1, \dots, X_n\}$ of degree $< p$ in each variable, by

$$\sigma = \sigma_1 \dots \sigma_n \leftrightarrow X_1^{\sigma_1} \dots X_n^{\sigma_n}.$$

Under this identification, we can write

$$\begin{aligned} V_n &= \mathbb{F}_p[X_1,\dots,X_n]/(X_1^p-1,\dots,X_n^p-1)\\ &= \mathbb{F}_p[X_1]/(X_1^p-1)\otimes\cdots\otimes\mathbb{F}_p[X_n]/(X_n^p-1). \end{aligned}$$

We write $g\sigma$ the action of $g\in G$ on $\sigma\in V_n$, and denote by $[g,\sigma]=\sigma-g\sigma$ the "Lie action" of G on V_n. For $r\in\{0,\dots,p^n-1\}$ we write $r=r_n\dots r_1$ in base p, and define

$$v_n^r=(1-X_1)^{r_1}\dots(1-X_n)^{r_n}\in V_n,$$
$$V_n^r=\langle v_n^r,\dots,v_n^{p^n-1}\rangle.$$

We extend the last definition to $V_n^r=0$ when $r\geq p^n$. There is a natural projective sequence

$$\cdots\to V_n\to V_{n-1}\to\cdots\to V_0=\mathbb{F}_p$$

of G-modules, and at each step n a sequence of V_n-submodules

$$V_n^{p^n}=0\subset\cdots\subset V_n^{p^n-p^{n-1}}\subset\cdots\subset V_n^1\subset V_n^0=V_n$$

each having codimension 1 in the next. Moreover $V_{n-1}^{p^{n-1}-i}$ and $V_n^{p^n-i}$ are naturally isomorphic under multiplication by $(1-X_n)^{p-1}$; thus $V_n^{p^n-p^{n-1}}$ is isomorphic to $V_{n-1}^0=V_{n-1}$ as a G-module.

Lemma D.17. 1. *The inclusion $[G,V_n^r]\subset V_n^{r+1}$ holds for all n and all r.*
2. *If G contains for all $m\leq n$ an element g_m such that*

$$g_m(0^m)=0^{m-1}1,\qquad g_m(\sigma x)=\sigma' x\quad\forall\sigma\in\Sigma^{m-1}\setminus\{0^{m-1}\},x\in\Sigma$$

(where in the second condition σ' is an arbitrary function of σ), then $[G,V_n^r]=V_n^{r+1}$ for all n and all r.

A G-module V having the property $\dim V^{(n)}/V^{(n+1)}=1$ for all n, where the $V^{(n)}$ are defined inductively by $V^{(0)}=V$ and $V^{(n+1)}=[G,V^{(n)}]$ is called *uniserial.* This notion was introduced by Leedham-Green [**LG94**]; see also [**DdSMS91**, page 111].

Note that every element of G can be described by a colouring $\{g_\sigma\}_{\sigma\in\Sigma^*}$ of the vertices of Σ^* by elements of the cyclic group $C_p=\langle\epsilon\rangle$. The condition in the lemma amounts to the existence, for all m, of an element g_m whose colouring is ϵ at the vertex 0^m, and is 1 on all other vertices of the m-th level as well as on all vertices 0^i, for $i<m$. Note also that this implies that the action is spherically transitive.

Proof. We proceed by induction on (n,r) in lexicographic order. For $n=0$ the claim holds trivially; suppose thus $n\geq 1$. In order to prove $[G,V_n^r]\subset V_n^{r+1}$, it suffices to check that for all $g\in G$ we have $[g,v_n^r]\in V_n^{r+1}$, as the V_n^r form an ascending tower of subspaces. During the proof we will consider V_{n-1} as a subspace of V_n; beware though that it is not a submodule. We shall write '$*$' for the action of G on $V_{n-1}\subset V_n$, and '$\cdot$' for that of G on V_n.

Observe that if $v\in V_{n-1}$ then $g\cdot(vX_n^i)=(g\cdot v)X_n^i$. Thus $g\cdot v-g*v$ is always divisible by $1-X_n$ because if $g\cdot v=\sum_{s=0}^{p-1}\Psi_sX_n^s$ for some $\Psi_s\in V_{n-1}$ then $g*v=\sum_{s=0}^{p-1}\Psi_s$ and

$$g\cdot v-g*v=(1-X_n)\sum_{s=1}^{p-1}-\Psi_s(1+X_n+\cdots+X_n^{s-1}).\tag{35}$$

Write $r=r_n\dots r_1$ in base p. For some Φ and Ψ_s in V_{n-1}, we may write

$$v_n^r=\Phi(1-X_n)^{r_n},\qquad g\cdot v_n^r=\sum_{s=0}^{p-1}\Psi_sX_n^s(1-X_n)^{r_n}.$$

Then by induction

$$[g, v_n^r] = \underbrace{\underbrace{\left(\Phi - \sum_{s=0}^{p-1} \Psi_s\right)}_{\in V_{n-1}^{(r+1) \bmod p^{n-1}}} (1 - X_n)^{r_n}}_{\in V_n^{r+1}} + \underbrace{\sum_{s=0}^{p-1} \Psi_s (1 - X_n^s)(1 - X_n)^{r_n}}_{\in V_n^{(r_n+1)p^{n-1}} \subseteq V_n^{r+1}},$$

as in the second summand $(1 - X_n^s)(1 - X_n)^{r_n}$ is divisible by $(1 - X_n)^{r_n+1}$. This proves the first claim of the lemma.

Next, we prove $[G, V_n^r] \supset V_n^{r+1}$ by showing that $v_n^{r+1} \in [G, V_n^r]$. As above, write $r = r_n \dots r_1$ in base p. If $(r_1, \dots, r_{n-1}) \neq (p-1, \dots, p-1)$, we have $v_n^{r+1} = v_{n-1}^{r+1 \bmod p^{n-1}}(1 - X_n)^{r_n}$, and by induction $v_{n-1}^{r+1 \bmod p^{n-1}} = \sum_s \alpha_s [g_s, v_{n-1}^{i_s}]$ for some $\alpha_s \in \mathbb{F}_p$, $g_s \in G$ and $i_s \geq r \bmod p^{n-1}$. Then

$$v_n^{r+1} = \underbrace{\sum_s \alpha_s \left[g_s, v_n^{i_s + r_n p^{n-1}}\right]}_{\in [G, V_n^r]} + \underbrace{\sum_s \alpha_s \left(g_s \cdot v_n^{i_s + r_n p^{n-1}} - (g_s * v_{n-1}^{i_s})(1 - X_n)^{r_n}\right)}_{\in V_n^{(r_n+1)p^{n-1}} \subseteq V_n^{r+2} \subseteq [G, V_n^{r+1}] \subseteq [G, V_n^r]}$$

where the last inclusions hold by (35) and induction. Finally, if $r = (r_n + 1)p^{n-1} - 1$, note that

$$\begin{aligned} v_n^r &= (1 + X_1 + \dots + X_1^{p-1}) \cdots (1 + X_{n-1} + \dots + X_{n-1}^{p-1})(1 - X_n)^{r_n} \\ &= (1 - X_n)^{r_n} + P(1 - X_n)^{r_n}, \end{aligned}$$

where $P = \sum_{\sigma \in \Sigma^{n-1} \setminus \{0^{n-1}\}} X_1^{\sigma_1} \cdots X_n^{\sigma_n}$ is invariant under g_n; thus

$$v_n^{r+1} = (1 - X_n)^{r_n} - X_n(1 - X_n)^{r_n} = [g_n, v_n^r] \in [G, V_n^r].$$

□

The strategy we follow to compute the lower central series or dimension series of G in the examples of Sections D.6 and D.7 is the following:

- We recognize some $\gamma_m(G)$ or G_m as a subgroup of G simply obtained from rigid stabilizers in G.
- We identify a quotient $\gamma_m(G)/N$ or G_m/N with a direct sum of copies of the module V_n defined above, for an appropriate subgroup N.
- We show that N is a further term of the lower central or dimensional series, allowing the process to repeat.

Then the exact terms of the lower central or dimension series are obtained by pulling back the appropriate V_n^r through the identification.

D.6. The Group $\mathfrak{G}$

Let $\Sigma = \mathbb{F}_2$, the field on two elements. For $x \in \mathbb{F}_2$ set $\overline{x} = 1 - x$, and define the automorphisms a, b, c, d of Σ^* as follows:

$$a(x\sigma) = \overline{x}\sigma,$$

$$\begin{aligned} b(0x\sigma) &= 0\overline{x}\sigma, & b(1\sigma) &= 1c(\sigma), \\ c(0x\sigma) &= 0\overline{x}\sigma, & c(1\sigma) &= 1d(\sigma), \\ d(0x\sigma) &= 0x\sigma, & d(1\sigma) &= 1b(\sigma). \end{aligned}$$

Thus for instance b acts on the subtree $0\Sigma^*$ as c, while c acts on it as d, etc. Note that all generators are of order 2 and $\{1, b, c, d\}$ forms a Klein group. Set $\mathfrak{G} = \langle a, b, c, d\rangle$. For ease of notation, we shall identify elements of $\mathsf{Stab}_{\mathfrak{G}}(n)$ with their image under ϕ_n by writing $\phi_n(g) = (g_1, \dots, g_{2^n})_n$ (omitting the subscript n if it is obvious from context); for instance we will write $b = (a, c)$, $c = (a, d)$ and $d = (1, b)$. Set $x = [a, b]$, and set

$$K = \langle x \rangle^{\mathfrak{G}} = \langle x, (x, 1), (1, x) \rangle.$$

Note that $(x, 1) = [b, d^a]$ and $(1, x) = [b^a, d]$. Also, K is a subgroup of finite index (actually index 16) in $\mathfrak{G}$, and contains $K \times K$ as a subgroup of finite index (actually index 4); for more details see [**Har00**] or [**BG99a**]. Set also $T = \langle x^2 \rangle^{\mathfrak{G}} = K^2$, and for any $Q \leq K$ define $Q_m = Q \times \cdots \times Q$ (2^m copies). Clearly $Q_m \leq \mathsf{Stab}_{\mathfrak{G}}(m)$ and acts on each subtree starting on level m by the corresponding factor. For $m \geq 1$ set $N_m = K_m \cdot T_{m-1}$.

For $m \geq 2$, we have $\mathsf{Rist}_{\mathfrak{G}}(m) = K_{m-2}$, so $\mathfrak{G}$ is a branch group.

LEMMA D.18. *The mapping*

$$\alpha \oplus \beta : N_m / N_{m+1} \longrightarrow V_m \oplus V_{m-1}$$

is an isomorphism for all m*, where the* V_m *are the modules defined in Subsection D.5.1,* α *maps* $(1, \dots, 1, x, 1, \dots, 1) \in K_m$ *to the monomial in* V_m *corresponding to the vertex at the* x*'s position, and* β *maps* $(1, \dots, 1, x^2, 1, \dots, 1) \in T_{m-1}$ *to the corresponding monomial in* V_{m-1}.

PROOF. We first suppose $m = 1$. Then $N_1/N_2 = \langle x^2, (1, x), (x, 1) \rangle / N_2$; it is easy to check that $x^4 = (x^2, x^2)$ modulo K_2, so all generators of N_1/N_2 are of order 2. Further, $[x^2, (1, x)] \in K_2$ and $[x^2, (x, 1)] \in K_2$, so the quotient N_1/N_2 is the elementary abelian group 2^3, and $\alpha \oplus \beta$ is an isomorphism in that case.

For $m > 1$ it suffices to note that both sides of the isomorphism are direct sums of 2^{m-1} terms on each of which the lemma for $m = 1$ can be applied. □

LEMMA D.19. *The following equalities hold in* $\mathfrak{G}$*:*

$$\begin{aligned} [x, a] &= x^2, & [x, b] &= x^2, \\ [x, c] &= x(1, x^{-1})x, & [x, d] &= (1, x), \\ [x^2, a] = x^4 &= ((U, V)x^2, (V, U)x^2), & [x^2, b] &= x^4, \\ [x^2, c] &= ((U, V)x^2, (1, x)), & [x^2, d] &= (1, (U, 1)x^2), \end{aligned}$$

where $U = (1, x^{-1})x$ *and* $V = (x^{-1}, 1)x^{-1}$ *are in* K.

PROOF. Direct computation; see also [**Roz96b**], where different notations are used. □

LEMMA D.20. *If* $Q \gneqq N_{m+1}$ *contains* $g = (x, \dots, x) \in K_m$*, then* $[Q, \mathfrak{G}] \geq N_{m+1}$.

PROOF. Let $b_m \in \{b, c, d\}$ be such that it acts like b on $1^m \Sigma^*$. Then

$$h = [g, b_m] = (1, \dots, 1, [x, b])_m = (1, \dots, 1, x^2)_m \in T_m.$$

Conjugating h by elements of g yields all cyclic permutations of the above vector, so as $[G, \mathfrak{G}]$ is normal in G it contains T_m. Likewise, let d_m act like d on $1^m \Sigma^*$. Then

$$[g, d_m] = (1, \dots, 1, [x, a], [x, d])_m = (1, \dots, 1, x^2, (1, x))_m;$$

using $T_m \leq [Q, \mathfrak{G}]$, we obtain $(1, \dots, 1, (1, x))_m = (1, \dots, 1, x)_{m+1} \in [Q, \mathfrak{G}]$, so by the same conjugation argument $[Q, \mathfrak{G}] \geq K_{m+1}$. □

THEOREM D.21. *For all $m \geq 1$ we have:*

(1) $\gamma_{2^m+1}(\mathfrak{G}) = N_m$.
(2) $\gamma_{2^m+1+r}(\mathfrak{G}) = N_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_{m-1}^r)$ *for* $r = 0, \dots, 2^m$.
(3)

$$\operatorname{rank}(\gamma_n(\mathfrak{G})/\gamma_{n+1}(\mathfrak{G})) = \begin{cases} 3 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 2 & \text{if } n = 2^m + 1 + r, \text{ with } 0 \leq r < 2^{m-1}, \\ 1 & \text{if } n = 2^m + 1 + r, \text{ with } 2^{m-1} \leq r \leq 2^m. \end{cases}$$

PROOF. First compute $\gamma_2(\mathfrak{G}) = \mathfrak{G}' = \langle [a, d], K\rangle$; it is of index 8 in $\mathfrak{G}$, with quotient generated by $\{a, b, c\}$. Compute also $\gamma_3(\mathfrak{G}) = \langle x^2 = [x, a], (1, x) = [x, d]\rangle^{\mathfrak{G}} = N_1$ of index 2 in $\gamma_2(\mathfrak{G})$, with quotient generated by $\{x^2, (1, x)\}$. This gives the basis of an induction on $m \geq 1$ and $0 < r \leq 2^m$.

Assume that $\gamma_{2^m+1}(\mathfrak{G}) = N_m$. Note that the hypothesis of Lemma D.17 is satisfied; indeed g_m can even be chosen among the conjugates of b, c or d. Consider the sequence of quotients $Q_r = N_{m+1}\gamma_{2^m+1+r}(\mathfrak{G})/N_{m+1}$ for $r \geq 0$. Lemmata D.18 and D.17 tell us that $Q_r = \alpha^{-1}(V_m^r) \oplus \beta^{-1}(V_{m-1}^r)$; in particular $Q_r \ni (x, \dots, x) = \alpha^{-1}(v_m^{2^m-1})$ for all $r < 2^m$, and then Lemma D.20 tells us that $\gamma_{2^m+1+r}(\mathfrak{G}) \geq N_{m+1}$ for $r \leq 2^m$. When $r = 2^m$ we have $\gamma_{2^{m+1}+1}(\mathfrak{G}) = N_{m+1}$ and the induction can continue. □

LEMMA D.22. *For all $m \geq 1$ and $r \in \{0, \dots, 2^m - 1\}$ we have:*

$$(\alpha^{-1}V_m^r)^2 = \beta^{-1}(V_m^r) \leq N_{m+1};$$
$$(\beta^{-1}V_{m-1}^r)^2 = \beta^{-1}(V_m^{r+2^{m-1}}) \leq N_{m+1}.$$

PROOF. Write $\alpha^{-1}(v_m^r) = (x^{i_1}, \dots, x^{i_{2^m}})$ or $\beta^{-1}(v_m^r) = (x^{2i_1}, \dots, x^{2i_{2^m}})$ for some $i_* \in \{0, 1\}$. Then these claims follow immediately, using Lemma D.19, from

$$\begin{aligned}(\alpha^{-1}v_m^r)^2 &= (x^{i_1}, \dots, x^{i_{2^m}})^2 = (x^{2i_1}, \dots, x^{2i_{2^m}}) = \beta^{-1}(v_m^r), \\ (\beta^{-1}v_{m-1}^r)^2 &= (x^{2i_1}, \dots, x^{2i_{2^{m-1}}})^2 = (x^{4i_1}, \dots, x^{4i_{2^{m-1}}}) \\ &\equiv (x^{2i_1}, x^{2i_1}, \dots, x^{2i_{2^{m-1}}}, x^{2i_{2^{m-1}}}) = \beta^{-1}(v_m^{r+2^{m-1}}) \mod N_{m+1}.\end{aligned}$$

□

THEOREM D.23. *For all $m \geq 1$ we have:*

(1) $\mathfrak{G}_{2^m+1} = N_m$.
(2)

$$\mathfrak{G}_{2^m+1+r} = \begin{cases} N_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_{m-1}^{r/2}) & \text{if } 0 \leq r \leq 2^m \text{ is even}, \\ N_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_{m-1}^{(r-1)/2}) & \text{if } 0 \leq r \leq 2^m \text{ is odd}. \end{cases}$$

(3)

$$\operatorname{rank}(\mathfrak{G}_i/\mathfrak{G}_{i+1}) = \begin{cases} 3 & \text{if } i = 1, \\ 2 & \text{if } i > 1 \text{ is even}, \\ 1 & \text{if } i > 1 \text{ is odd}. \end{cases}$$

PROOF. First compute $\mathfrak{G}_2 = \gamma_2(\mathfrak{G})$ and $\mathfrak{G}_3 = \gamma_3(\mathfrak{G}) = N_1$. This gives the basis of an induction on $m \geq 1$ and $0 \leq r \leq 2^m$. Assume $\mathfrak{G}_{2^m+1} = N_m$. Consider the sequence of quotients $Q_{m,r} = N_{m+1}\mathfrak{G}_{2^m+1+r}/N_{m+1}$ for $r \geq 0$. We have $Q_{m,r} = [\mathfrak{G}, Q_{m,r-1}]Q^2_{m-1,\lfloor r/2\rfloor}$ by (32). Lemmata D.18, D.22 and D.17 tell us that $Q_r = \alpha^{-1}(V_m^r) \oplus \beta^{-1}(V_{m-1}^{\lfloor r/2\rfloor})$; in particular $Q_r \ni (x, \dots, x) = \alpha^{-1}(v_m^{2^m-1})$ for all $r < 2^m$, and then Lemma D.20 tells us that $\mathfrak{G}_{2^m+1+r} \geq N_{m+1}$ for $r \leq 2^m$. When $r = 2^m$ we have $\mathfrak{G}_{2^{m+1}+1} = N_{m+1}$ and the induction can continue. □

D.6.1. Cayley graphs of Lie algebras. We introduce the notion of *Cayley graph* for graded Lie algebras. Let $L = \bigoplus_{n=1}^{\infty} L_n$ be a graded Lie algebra generated by a finite set S of degree one elements. Fix a basis $(\ell_{n,1}, \dots, \ell_{n,\dim L_n})$ of L_n for every n, and give each L_n an orthogonal scalar product $\langle \ell_{n,i} | \ell_{n,j} \rangle = \delta_{i,j}$. The *Cayley graph* of L is defined as follows: its vertices are the $(i,j) \in \mathbb{N}^2$ with $i \geq 1$ and $1 \leq j \leq \dim L_n$. For every $s \in S$ and $i,j,k \in \mathbb{N}$ there is an edge from (i,j) to $(i+1,k)$ labeled by s and with weight $\langle [\ell_{i,j}, s] | \ell_{i+1,k} \rangle$. By convention edges of weight 0 are not represented. Additionally, if L is a p-algebra, there is an unlabeled edge of length $(p-1)i$ from (i,j) to (pi,k) with weight $\langle \ell_{i,j}^p | \ell_{pi,k} \rangle$.

Clearly, the Cayley graph of a Lie algebra L determines the structure of L. It is a connected graph, because S is a generating set. The geometric growth of the graph is the same as the growth of the algebra.

As a simple example, consider the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ generated by $\{i,j\}$, and its dimension series $Q_1 = Q$, $Q_2 = \{\pm 1\}$ and $Q_3 = 1$ over the field $\mathbb{F}_2$. Then the Cayley graph of $\mathcal{L}(Q)$ is

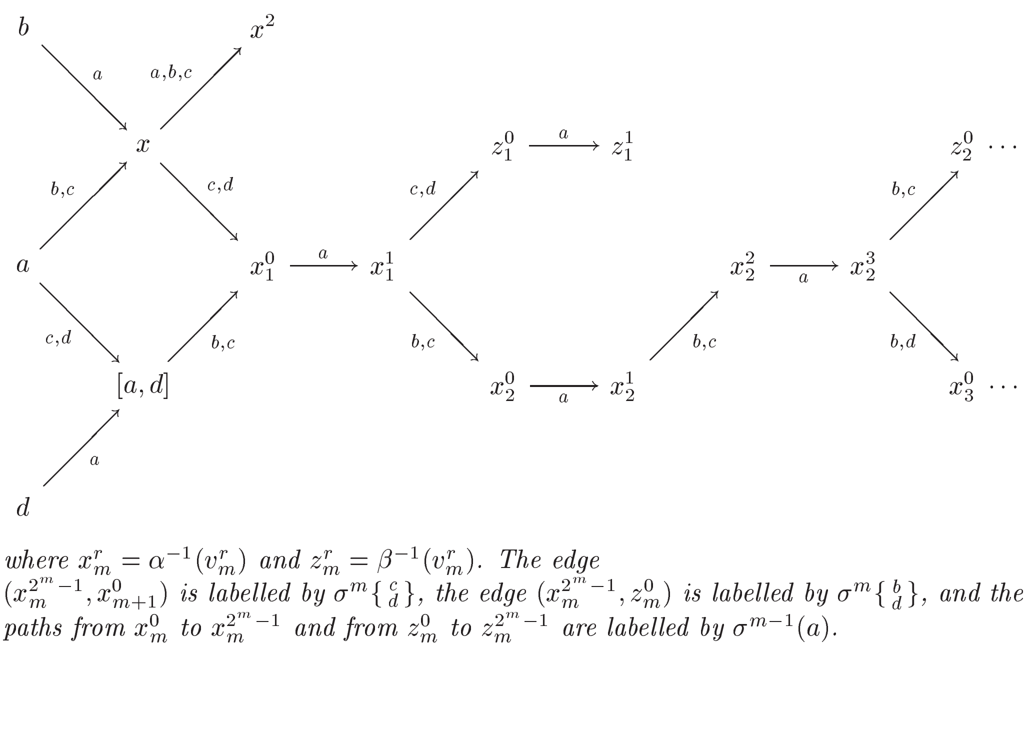
$$\begin{tikzcd} i \arrow[dr, "j"] & \\ & -1 \\ j \arrow[ur, "i"'] & \end{tikzcd}$$

We now describe the Cayley graphs of L and $\mathcal{L}_{\mathbb{F}_2}$ associated respectively to the lower central and dimensional series of $\mathfrak{G}$. Fix $S = \{a,b,c,d\}$ as a generating set for $\mathfrak{G}$, and extend it to $\overline{S} = \{a,b,c,d,\{{}^b_c\},\{{}^b_d\},\{{}^c_d\}\}$. Define the transformation σ on $\overline{S}^*$ by

$$\sigma(a) = a\{{}^b_c\}a, \quad \sigma(b) = d, \sigma(c) = b, \sigma(d) = b,$$

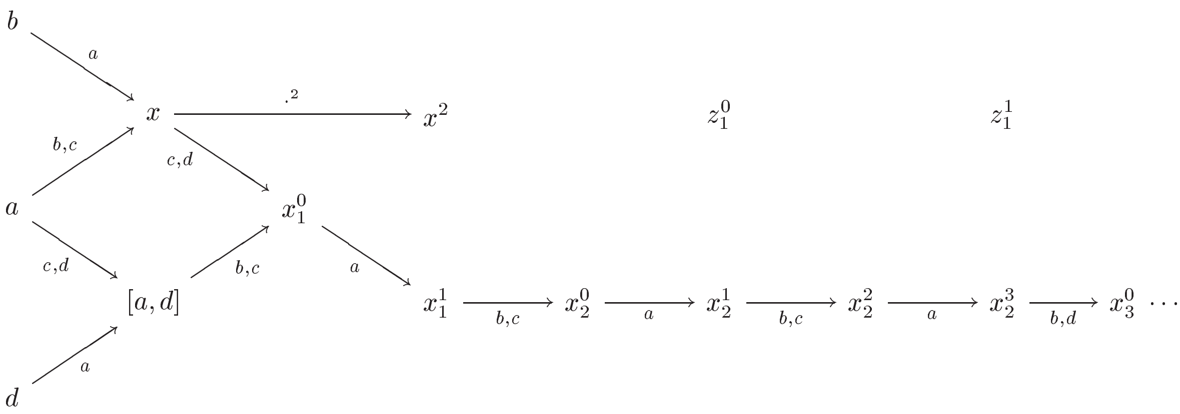
naturally extended to subsets. (For any fixed $g \in G$, one may obtain all elements $h \in \mathsf{Stab}_{\mathfrak{G}}(1)$ with $\phi(h) = (g,*)$ by computing $\sigma(g)$ and making all possible choices of a letter from the braced symbols. This explains the definition of $\overline{S}$.)

THEOREM D.24. *The Cayley graph of $L(\mathfrak{G})$ is as follows:*



where $x_m^r = \alpha^{-1}(v_m^r)$ and $z_m^r = \beta^{-1}(v_m^r)$. The edge $(x_m^{2^m-1}, x_{m+1}^0)$ is labelled by $\sigma^m\{{}^c_d\}$, the edge $(x_m^{2^m-1}, z_m^0)$ is labelled by $\sigma^m\{{}^b_d\}$, and the paths from x_m^0 to $x_m^{2^m-1}$ and from z_m^0 to $z_m^{2^m-1}$ are labelled by $\sigma^{m-1}(a)$.

The Cayley graph of $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$ *is as follows:*



with the same rule for labellings as for $L(\mathfrak{G})$*; and power maps from* x_m^r *to* z_m^r*.*

Note that as we are in characteristic 2 the non-zero weights can only be 1 and thus are not indicated.

D.7. The Group $\widetilde{\mathfrak{G}}$

We describe here the lower central and dimension series for a group $\widetilde{\mathfrak{G}}$ containing the previous section's group $\mathfrak{G}$ as a subgroup. More details about $\widetilde{\mathfrak{G}}$ can be found in [**BG99a**].

As in Section D.6 set $\Sigma = \mathbb{F}_2$, and define automorphisms $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ of Σ^* by

$$\tilde{b}(0x\sigma) = 0\overline{x}\sigma, \qquad \tilde{b}(1\sigma) = 1\tilde{c}(\sigma),$$
$$\tilde{c}(0\sigma) = 0\sigma, \qquad \tilde{c}(1\sigma) = 1\tilde{d}(\sigma),$$
$$\tilde{d}(0\sigma) = 0\sigma, \qquad \tilde{d}(1\sigma) = 1\tilde{b}(\sigma).$$

Note that all generators are of order 2 and $\{\tilde{b}, \tilde{c}, \tilde{d}\}$ generate the elementary abelian group 2^3. Set $\widetilde{\mathfrak{G}} = \langle a, \tilde{b}, \tilde{c}, \tilde{d}\rangle$. Clearly, $\mathfrak{G} = \langle a, b = \tilde{b}\tilde{c}, c = \tilde{c}\tilde{d}, d = \tilde{d}\tilde{b}\rangle$ is a subgroup of $\widetilde{\mathfrak{G}}$. Its index is infinite, because $\mathfrak{G}$ is a torsion group while $w = a\tilde{b}\tilde{c}\tilde{d}$ has infinite order, because $w^2 = (w^a, w)$. Set $x = [a, \tilde{b}]$, $y = [a, \tilde{d}]$, and

$$\tilde{K} = \langle x, y\rangle^{\widetilde{\mathfrak{G}}}.$$

Then $\tilde{K}$ is a subgroup of finite index (actually index 32) in $\mathfrak{G}$, and contains $\tilde{K} \times \tilde{K}$ as a subgroup of finite index (actually index 8). Set also $\tilde{T} = \langle x^2\rangle^{\mathfrak{G}} = \tilde{K}^2$, and for any $Q \le \tilde{K}$ define $Q_m = Q \times \cdots \times Q$ (2^m copies). For $m \ge 1$ set $\tilde{N}_m = \tilde{K}_m \cdot \tilde{T}_{m-1}$.

For $m \ge 2$, we have $\mathsf{Rist}_{\widetilde{\mathfrak{G}}}(m) = \tilde{K}_{m-2}$, so $\widetilde{\mathfrak{G}}$ is a branch group.

Lemma D.25. *The mapping*

$$\alpha \oplus \beta \oplus \gamma : \tilde{N}_m/\tilde{N}_{m+1} \longrightarrow V_m \oplus V_m \oplus V_{m-1}$$

is an isomorphism for all m*, where the* V_m *are the modules defined in Subsection D.5.1,* α *maps* $(1, \dots, x, \dots, 1) \in \tilde{K}_m$ *to the monomial in* V_m *corresponding to the vertex in* x*'s position, and* β *maps* $(1, \dots, y, \dots, 1) \in \tilde{K}_m$ *to the corresponding vertex in* V_m*, and* γ *maps* $(1, \dots, x^2, \dots, 1) \in \tilde{T}_{m-1}$ *to the corresponding monomial in* V_{m-1}*.*

Proof. We first suppose $m = 1$. Then

$$\tilde{N}_1/\tilde{N}_2 = \langle x^2, (1, x), (x, 1), (1, y), (y, 1)\rangle/\tilde{N}_2;$$

it is easy to check that $x^4 = 1$, so all generators of $\tilde{N}_1/\tilde{N}_2$ are of order 2. Further, all commutators of generators belong to $\tilde{K}_2$, so the quotient $\tilde{N}_1/\tilde{N}_2$ is the elementary abelian group 2^5, and $\alpha \oplus \beta \oplus \gamma$ is an isomorphism in that case.

For $m > 1$ it suffices to note that both sides of the isomorphism are direct sums of 2^{m-1} terms on each of which the lemma for $m = 1$ can be applied. $\square$

LEMMA D.26. *The following equalities hold in $\widetilde{\mathfrak{G}}$:*

$$\begin{aligned} [x,a] &= x^2, & [x,\tilde{b}] &= x^2 \\ [x,\tilde{c}] &= (1,y), & [x,\tilde{d}] &= (1,x), \\ [x^2,a] &= 1, & [x^2,\tilde{b}] &= 1, \\ [x^2,\tilde{c}] &= 1, & [x^2,\tilde{d}] &= (1,x(x,1)x), \\ [y,a] &= 1, & [y,\tilde{b}] &= (x^{-1},1), \\ [y,\tilde{c}] &= 1, & [y,\tilde{d}] &= 1. \end{aligned}$$

PROOF. Direct computation. $\square$

LEMMA D.27. *If $Q \gneqq \tilde{N}_{m+1}$ contains $g = (x,\dots,x) \in \tilde{K}_m$, then $[Q,\widetilde{\mathfrak{G}}] \geq \tilde{N}_{m+1}$.*

PROOF. Let $\tilde{b}_m \in \{\tilde{b},\tilde{c},\tilde{d}\}$ be such that it acts like $\tilde{b}$ on $1^m\Sigma^*$. Then

$$[g,\tilde{b}_m] = (1,\dots,1,[x,\tilde{b}])_m = (1,\dots,1,x^2)_m \in \tilde{T}_m,$$

so by a conjugation argument $[Q,\widetilde{\mathfrak{G}}] \geq \tilde{T}_m$. Likewise, let $\tilde{c}_m$ and $\tilde{d}_m$ act like c and d on $1^m\Sigma^*$. Then

$$\begin{aligned} [g,\tilde{c}_m] &= (1,\dots,1,[x,a],[x,\tilde{c}])_m = (1,\dots,1,x^2,(1,y))_m, \\ [g,\tilde{d}_m] &= (1,\dots,1,(1,x))_m. \end{aligned}$$

Using $\tilde{T}_m \leq [Q,\widetilde{\mathfrak{G}}]$, we obtain $(1,\dots,1,(1,y))_m = (1,\dots,1,y)_{m+1} \in [Q,\widetilde{\mathfrak{G}}]$, so again by a conjugation argument $[Q,\widetilde{\mathfrak{G}}] \geq \tilde{K}_{m+1}$. $\square$

THEOREM D.28. *For all $m \geq 1$ we have:*

(1) $\gamma_{2^m+1}(\widetilde{\mathfrak{G}}) = \tilde{N}_m$.
(2) $\gamma_{2^m+1+r}(\widetilde{\mathfrak{G}}) = \tilde{N}_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_m^r)\gamma^{-1}(V_{m-1}^r)$ *for* $r = 0,\dots,2^m$.
(3)

$$\operatorname{rank}(\gamma_n(\widetilde{\mathfrak{G}})/\gamma_{n+1}(\widetilde{\mathfrak{G}})) = \begin{cases} 4 & \text{if } n = 1, \\ 3 & \text{if } n = 2, \\ 3 & \text{if } n = 2^m+1+r, \text{ with } 0 \leq r < 2^{m-1}, \\ 2 & \text{if } n = 2^m+1+r, \text{ with } 2^{m-1} \leq r \leq 2^m. \end{cases}$$

PROOF. First compute $\gamma_2(\widetilde{\mathfrak{G}}) = \widetilde{\mathfrak{G}}' = \langle [a,\tilde{c}], \tilde{K} \rangle$, of index 16 in $\widetilde{\mathfrak{G}}$, and $\gamma_3(\widetilde{\mathfrak{G}}) = \langle x^2,(1,x),(1,y)\rangle^{\widetilde{\mathfrak{G}}} = \tilde{N}_1$, with $x^2 = [x,a]$, $(1,x) = [x,\tilde{d}]$ and $(1,y) = [x,\tilde{c}]$. This gives the basis of an induction on $m \geq 1$ and $0 \leq r \leq 2^m$.

Assume that $\gamma_{2^m+1}(\widetilde{\mathfrak{G}}) = \tilde{N}_m$. Note that the hypothesis of Lemma D.17 is satisfied for $\widetilde{\mathfrak{G}}$, as it holds for $\mathfrak{G} < \widetilde{\mathfrak{G}}$. Consider the sequence of quotients $Q_r = \tilde{N}_{m+1}\gamma_{2^m+1+r}(\widetilde{\mathfrak{G}})/\tilde{N}_{m+1}$ for $r \geq 0$. Lemmata D.25 and D.17 tell us that $Q_r = \alpha^{-1}(V_m^r) \oplus \beta^{-1}(V_m^r) \oplus \gamma^{-1}(V_{m-1}^r)$; in particular $Q_r \ni (x,\dots,x) = \alpha^{-1}(v_m^{2^m-1})$ for all $r < 2^m$, and then Lemma D.27 tells us that $\gamma_{2^m+1+r}(\widetilde{\mathfrak{G}}) \geq \tilde{N}_{m+1}$ for $r \leq 2^m$. When $r = 2^m$ we have $\gamma_{2^{m+1}+1}(\widetilde{\mathfrak{G}}) = \tilde{N}_{m+1}$ and the induction can continue. $\square$

LEMMA D.29. *For all $m \geq 1$ and $r \in \{0,\dots,2^m-1\}$ we have:*

$$\begin{aligned} (\alpha^{-1}V_m^r)^2 &= \gamma^{-1}(V_m^r) \leq \tilde{N}_{m+1}; \\ (\beta^{-1}V_m^r)^2 &= 1 \leq \tilde{N}_{m+1}; \\ (\gamma^{-1}V_{m-1}^r)^2 &= \gamma^{-1}(V_m^{r+2^{m-1}}) \leq \tilde{N}_{m+1}. \end{aligned}$$

PROOF. Write $\alpha^{-1}(v_m^r) = (x^{i_1}, \dots, x^{i_{2^m}})$, $\beta^{-1}(v_m^r) = (y^{i_1}, \dots, y^{i_{2^m}})$ or $\gamma^{-1}(v_m^r) = (x^{2i_1}, \dots, x^{2i_{2^m}})$ for some $i_* \in \{0,1\}$. Then these claims follow immediately, using Lemma D.26, from

$$\begin{aligned}
(\alpha^{-1}v_m^r)^2 &= (x^{i_1}, \dots, x^{i_{2^m}})^2 = (x^{2i_1}, \dots, x^{2i_{2^m}}) = \gamma^{-1}(v_m^r),\\
(\beta^{-1}v_m^r)^2 &= (y^{i_1}, \dots, y^{i_{2^m}})^2 = (y^{2i_1}, \dots, y^{2i_{2^m}}) = (1, \dots, 1),\\
(\gamma^{-1}v_{m-1}^r)^2 &= (x^{2i_1}, \dots, x^{2i_{2^{m-1}}})^2 = (x^{4i_1}, \dots, x^{4i_{2^{m-1}}})\\
&\equiv (x^{2i_1}, x^{2i_1}, \dots, x^{2i_{2^{m-1}}}, x^{2i_{2^{m-1}}}) = \gamma^{-1}(v_m^{r+2^{m-1}}) \mod \tilde{N}_{m+1}.
\end{aligned}$$

□

THEOREM D.30. *For all $m \geq 1$ we have:*

(1) $\widetilde{\mathfrak{G}}_{2^m+1} = \tilde{N}_m$.
(2)

$$\widetilde{\mathfrak{G}}_{2^m+1+r} = \begin{cases} \tilde{N}_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_m^r)\gamma^{-1}(V_{m-1}^{r/2}) & \text{if } 0 \leq r \leq 2^m \text{ is even,}\\ \tilde{N}_{m+1}\alpha^{-1}(V_m^r)\beta^{-1}(V_m^r)\gamma^{-1}(V_{m-1}^{(r-1)/2}) & \text{if } 0 \leq r \leq 2^m \text{ is odd.}\end{cases}$$

(3)

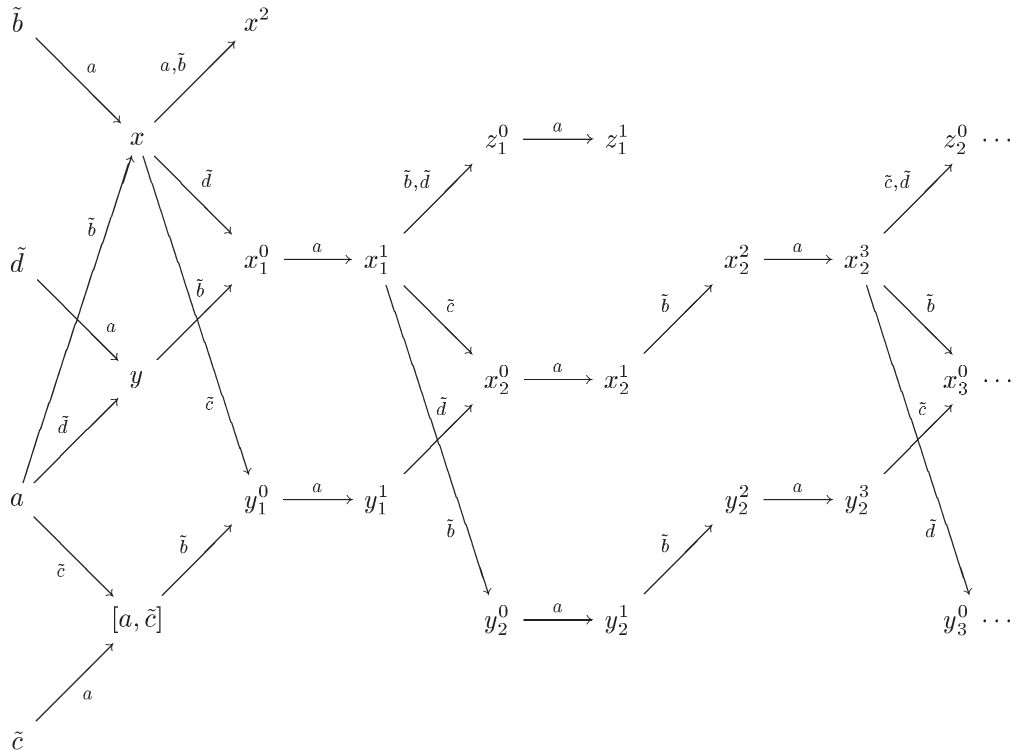
$$\operatorname{rank}(\widetilde{\mathfrak{G}}_i/\widetilde{\mathfrak{G}}_{i+1}) = \begin{cases} 4 & \text{if } i = 1,\\ 3 & \text{if } i > 1 \text{ is even,}\\ 2 & \text{if } i > 1 \text{ is odd.}\end{cases}$$

PROOF. First compute $\widetilde{\mathfrak{G}}_2 = \gamma_2(\widetilde{\mathfrak{G}})$ and $\widetilde{\mathfrak{G}}_3 = \gamma_3(\widetilde{\mathfrak{G}}) = \tilde{N}_1$. This gives the basis of an induction on $m \geq 1$ and $0 \leq r \leq 2^m$. Assume $\mathfrak{G}_{2^m+1} = \tilde{N}_m$. Consider the sequence of quotients $Q_{m,r} = \tilde{N}_{m+1}\widetilde{\mathfrak{G}}_{2^m+1+r}/\tilde{N}_{m+1}$ for $r \geq 0$. We have $Q_{m,r} = [\widetilde{\mathfrak{G}}, Q_{m,r-1}]Q^2_{m-1,\lfloor r/2\rfloor}$ by (32). Lemmata D.25, D.29 and D.17 tell us that $Q_r = \alpha^{-1}(V_m^r) \oplus \beta^{-1}(V_m^r) \oplus \gamma^{-1}(V_{m-1}^{\lfloor r/2\rfloor})$; in particular $Q_r \ni (x, \dots, x) = \alpha^{-1}(v_m^{2^m-1})$ for all $r < 2^m$, and then Lemma D.27 tells us that $\mathfrak{G}_{2^m+1+r} \geq \tilde{N}_{m+1}$ for $r \leq 2^m$. When $r = 2^m$ we have $\mathfrak{G}_{2^{m+1}+1} = \tilde{N}_{m+1}$ and the induction can continue. □

D.7.1. The Lie Algebra Structures. We describe here the Cayley graphs of L and $\mathcal{L}_{\mathbb{F}_p}$ associated respectively to the lower central and dimension series of $\widetilde{\mathfrak{G}}$. Consider $\tilde{S} = \{a, \tilde{b}, \tilde{c}, \tilde{d}\}$ and define the transformation $\tilde{\sigma}$ on $\tilde{S}^*$ by

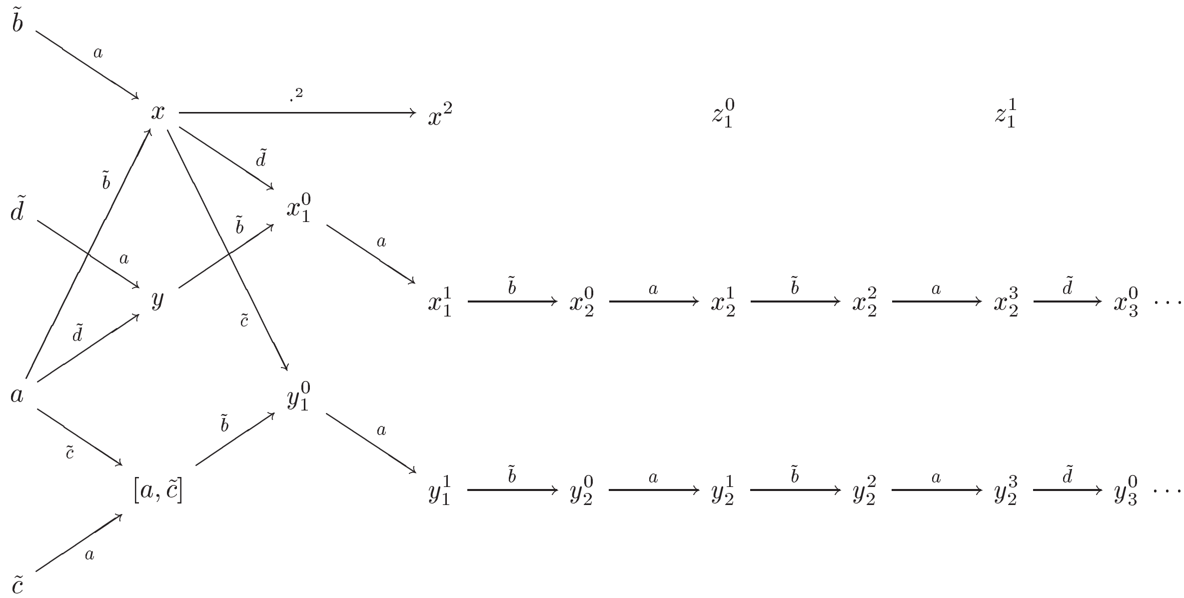
$$\tilde{\sigma}(a) = a\tilde{b}a, \quad \tilde{\sigma}(\tilde{b}) = \tilde{d}, \quad \tilde{\sigma}(\tilde{c}) = \tilde{b}, \quad \tilde{\sigma}(\tilde{d}) = \tilde{b}.$$

THEOREM D.31. *The Cayley graph of $L(\widetilde{\mathfrak{G}})$ is as follows:*



where $x_m^r = \alpha^{-1}(v_m^r)$, $y_m^r = \beta^{-1}(v_m^r)$ and $z_m^r = \gamma^{-1}(v_m^r)$. The edge $(x_m^{2^m-1}, x_{m+1}^0)$ is labelled by $\tilde{\sigma}^m(\tilde{d})$, the edges $(x_m^{2^m-1}, y_{m+1}^0)$ and $(x_m^{2^m-1}, z_m^0)$ are labelled by $\tilde{\sigma}^m(\tilde{b})$, the edges $(x_m^{2^m-1}, z_m^0)$ and $(y_m^{2^m-1}, x_{m+1}^0)$ are labelled by $\tilde{\sigma}^m(\tilde{c})$, and the paths from x_m^0 to $x_m^{2^m-1}$, from y_m^0 to $y_m^{2^m-1}$ and from z_m^0 to $z_m^{2^m-1}$ are labelled by $\tilde{\sigma}^{m-1}(a)$.

The Cayley graph of $\mathcal{L}_{\mathbb{F}_2}(\widetilde{\mathfrak{G}})$ is as follows:



with the same labellings as for $L(\widetilde{\mathfrak{G}})$; and power maps from x_m^r to z_m^r.

D.8. Other Fractal Groups

The technique involved in the proof of the results of the last three sections show that for a group G acting on a tree Σ^* by powers of the cyclic permutation $\epsilon = (0, 1, \dots, p-1)$ at each vertex, G has finite width when the following conditions are satisfied:

(1) the corresponding action on a sequence $\{V_n\}_{n=0}^{\infty}$ of G-modules as defined in Subsection D.5.1 has the *bounded corank property*, i.e. there is a constant C such that
$$\dim V_n^r/[G, V_n^r] \leq C$$
for all $n \geq 0$ and $0 \leq r \leq p^n - 1$.

(2) There is a descending sequence $\{N_m\}_{m=1}^{\infty}$ of normal subgroups of G satisfying the condition that for all m the quotients N_m/N_{m+1} are isomorphic to some direct sum $\bigoplus_{i=1}^{K} V_{m+\delta_i}$ for fixed K and δ_i.

Let us mention that the p-groups G_ω, for arbitrary $p \geq 2$ and $\omega \in \{0, \dots, p\}^{\mathbb{N}}$ constructed in [**Gri84, Gri85**] all satisfy Condition 1. Also, the group $\langle a, t\rangle < \mathsf{Aut}(\Sigma_p^*)$, $p \geq 3$, where a acts as ϵ on the root vertex and trivially elsewhere and t is defined recursively by $t = (a, 1, \dots, 1, t)$, satisfies Condition 1. We believe that this last group also satifies Condition 2, as do all G_ω for periodic sequences ω. Note that $\mathfrak{G}$ is a particular case of G_ω when $p = 2$ and $\omega = 012012\dots$. Therefore they all 'should' have finite width.

Meanwhile, the Gupta-Sidki groups constructed in [**GS83a**] do not satisfy Condition 1. As was proved recently by the first author, the growth of the Lie algebra $\mathcal{L}_{\mathbb{F}_p}(G)$ coincides with the spherical growth of the Schreier graph of G relatively to $\mathsf{Stab}_G(e)$, where e is an infinite geodesic path in the tree Σ^*. For our groups $\mathfrak{G}$ and $\widetilde{\mathfrak{G}}$ the spherical growth is bounded and this is why these groups have bounded width. For the Gupta-Sidki groups, the spherical growth of the Schreier graph is unbounded (it grows approximately as $\sqrt{n}$), and therefore these groups do not have the finite width property. It also follows from these considerations that their growth is at least $e^{n^{1-1/(1/2+2)}} = e^{n^{3/5}}$.

D.9. Profinite Groups of Finite Width

Finally we wish to explain how our results in the previous sections lead to counterexamples to Conjecture D.1 stated in the introduction. Let $\widehat{G}$ be the profinite completion of $G = \mathfrak{G}$ or $\widetilde{\mathfrak{G}}$.

THEOREM D.32. *The group $\widehat{G}$ is a just-infinite pro-2-group of finite width which does not belong to the list of Conjecture D.1 (which consists of solvable groups, p-adic analytic groups, and groups commensurable to positive parts of loop groups or to the Nottingham group).*

Its proof relies on the following notion:

DEFINITION D.33. *Let $G < \mathsf{Aut}(\Sigma^*)$ be a group acting on a rooted tree. G has the* congruence subgroup property *if for any finite-index subgroup H of G there is an n such that $\mathsf{Stab}_G(n) < H < G$.*

PROOF. G has the congruence property. This is well known for $\mathfrak{G}$ (see for instance [**Gri00**]); while for $\widetilde{\mathfrak{G}}$ the subgroup $\tilde{K}$ contains $\mathsf{Stab}_{\widetilde{\mathfrak{G}}}(4)$ and enjoys the property that every subgroup of finite index in $\widetilde{\mathfrak{G}}$ contains $\tilde{K}_m = \tilde{K} \times \cdots \times \tilde{K}$ for some m; see [**BG99a**].

The profinite completion of G with respect to its subgroups $\mathsf{Stab}_G(n)$ is therefore a pro-2-group and coincides with the closure of G in $\mathsf{Aut}(\{0,1\}^*)$. The closure of a branch group is again a branch group, as is observed in [**Gri00**].

The criterion of just-infiniteness for profinite branch groups is the same as the one for discrete branch groups given in [**BG99a**]; it is that $\overline{K}/\overline{K}'$ (respectively $\overline{\tilde{K}}/\overline{\tilde{K}}'$) be finite, where $\overline{K}$ and $\overline{\tilde{K}}$ are the closures of K and $\tilde{K}$. Now $|\overline{K}/\overline{K}'| \leq |K/K'| < \infty$, the last inequality following from a computation in [**BG99a**]. The same inequalities hold for $\widetilde{\mathfrak{G}}$, and this proves the just-infiniteness of $\widehat{G}$.

The group $\widehat{G}$ has finite width for both versions of Definition D.2. This is clear for D-width, because the discrete and pro-p Lie algebras $\mathcal{L}(G)$ and $\mathcal{L}(\widehat{G})$ are isomorphic. The finiteness of C-width follows from the inequalities

$$\left|\gamma_n(\widehat{G})/\gamma_{n+1}(\widehat{G})\right| \leq |\gamma_n(G)/\gamma_{n+1}(G)| < \infty,$$

which again are consequences of the congruence property of G.

Finally, $\widehat{G}$ does not belong to the list of groups given in Conjecture D.1: it is neither solvable, because G isn't, nor p-adic analytic, by Lazard's criterion [**Laz65**] (its Lie algebra $\mathcal{L}(\widehat{G}) = \mathcal{L}(G)$ would have a zero component in some dimension). The other groups in the list of Conjecture D.1 are *hereditarily just-infinite* groups, that is, groups every open subgroup of which is just-infinite [**KLP97**, page 5]. Profinite just-infinite branch groups are never hereditarily just-infinite, as is shown in [**Gri00**]. □

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ANNEXE E

On the Spectrum of Hecke Type Operators related to some Fractal Groups

Laurent Bartholdi and Rostislav I. Grigorchuk

E.1. Introduction

The Hecke, Markov, and Laplace operators occur in various guises throughout mathematics. We start by a review of their more common appearances.

E.1.1. Discrete Laplacian and Hecke type Operators. Let $\mathfrak{G} = (V, E)$ be a locally finite graph: there are maps $\alpha, \omega : E \to V$ giving the extremities of edges, and every vertex $v \in V$ has finite degree $\deg v = |\{e \in E | \alpha(e) = v\}|$. Therefore all edges are oriented, and $\mathfrak{G}$ may have loops ($\alpha(e) = \omega(e)$) and multiple edges. The *discrete Laplace operator* of $\mathfrak{G}$ is the operator $\Delta = 1 - M$ on $\ell^2(V, \deg)$, where M is the "adjacency" or Markovian operator

$$(Mf)(v) = \frac{1}{\deg v} \sum_{e \in E\colon \alpha(e) = v} f(\omega(e)).$$

The theory of discrete Laplace operators Δ has a long history, and is a popular topic of contemporary mathematics [**CDS79, Woe94**]. In the context of random walks on graphs, one usually considers the Markovian operator M rather than Δ: if e_v be the Dirac delta function at the vertex v, then $\langle M^n e_v | e_w \rangle$ is the probability of a random walk starting at v to reach w in n steps. If the random walk is symmetric, then M is a self-adjoint operator and its spectrum lies in $[-1, 1]$. The spectral properties of M contain valuable information for the theory of random walks and discrete potential, graph theory, abstract harmonic analysis, the theory of operator algebras, etc. For instance, a theorem by Harry Kesten [**Kes59**], generalized by different mathematicians (see [**Woe94**] and [**HGC99**] with its bibliography) asserts that $\mathfrak{G}$ is amenable if and only if 1 is in the spectrum of M. Note that the random walk need not be simple (probabilities of moving in different directions may be different); a Markovian operator can still be associated to the walk. Let us finally mention a more general setting, developed these last years [**Nov97**]: the (discrete) Schrödinger operators $\Delta + P$, where P is diagonal, and the coefficients of Δ may depend on the vertex they correspond to.

The theory of Hecke type operators was developed in parallel: if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of a finitely generated group G given with a symmetric generating system $S = \{s_1, \dots, s_m\} = S^{-1}$ in a Hilbert space $\mathcal{H}$, then one associates to π a self-adjoint operator H on $\mathcal{H}$:

$$H = \sum_{i=1}^{m} p(i) \pi(s_i),$$

for some $p(i) \in \mathbb{C}$. The most important choice is $p(i) = \frac{1}{m}$ for all $i \in \{1, \dots, m\}$; we shall restrict to this choice in the sequel, and assume, when no weight is given, that this one is used.

Operators of Hecke type play an important role in mathematical physics [**Con94**], Arakelov theory in number theory (see [**Li96**], [**Ser97**] and [**Ser95**] for the connection between number theory and operators), and Ramanujan graphs [**Lub94**]. The group-theoretical content of H, mainly in the case of the regular representation, was studied by Pierre de la Harpe, A. Guyan Robertson and Alain Valette in [**HRV93a, HRV93b**], and in many other papers — see the bibliography in [**Woe94**].

E.1.2. Spectra of Noncommutative Dynamical Systems. Let T be an invertible measure-preserving transformation of a measure space (X,μ), and let A be the corresponding unitary operator in $L^2(X,\mu)$, given by $(Af)(x) = f(T^{-1}x)$. By the *spectrum* of the dynamical system one usually means the spectrum of the operator A, or, as is almost the same, the spectrum of the self-adjoint operator $A + A^{-1}$. These last spectra are in correspondence through the map $z \mapsto z + z^{-1}$. If T is aperiodic, then $\operatorname{spec}(A + A^{-1}) = [-1,1]$ (see for instance [**Pau99**, Proposition 1.18]).

By a *noncommutative dynamical system* we mean a collection S of invertible measure-class-preserving transformations on a measure space (X,μ), that do not necessarily commute. Let G be the group generated by S. It has a natural unitary representation π in $\mathcal{L}^2(X,\mu)$ given by

$$(\pi(g)f)(x) = \sqrt{\mathfrak{g}(x)}f(g^{-1}x),$$

where $\mathfrak{g}(x) = dg\mu(x)/d\mu(x)$ is the Radon-Nikodým derivative. The *spectrum* of the dynamical system S is the spectrum of the Hecke type operator associated to G, $S \cup S^{-1}$ and π. In case $|S| = 1$, this definition reduces to the previous, classical one.

E.1.3. Examples. One of the most famous operators of Hecke type is the Harper-Mathieu-Peierls operator H_λ on $\ell^2(\mathbb{Z})$ acting on infinite sequences $f : \mathbb{Z} \to \mathbb{C}$ by

$$(H_\lambda(f))(n) = f(n-1) + f(n+1) + (2\cos\lambda n)f(n),$$

for any $\lambda \in \mathbb{R}$. These H_λ are the Hecke type operators associated to the Heisenberg group

$$\left\{ \begin{pmatrix} 1 & m & p \\ & 1 & n \\ & & 1 \end{pmatrix} \middle|\ m,n,p \in \mathbb{Z} \right\} \cong \langle a,b,c |\, [a,b] = c, [a,c] = [b,c] = 1 \rangle$$

and its representation π_λ in $\ell^2(\mathbb{Z})$, where $\pi_\lambda(a)$ acts by translation: $[\pi_\lambda(a)](f)(n) = f(n-1)$, and $\pi_\lambda(b)$ acts by pointwise multiplication with the function $e^{i\lambda n}$, namely $[\pi_\lambda(b)](f)(n) = e^{i\lambda n}f(n)$.

The Harper operator is the operator related to the Quantum Hall effect, and originally arose in connection with the two-dimensional lattice. One can start from any Cayley graph, and construct a corresponding Harper operator, which would be the discrete analogue of the magnetic Laplacian [**CHMM98**].

The spectral properties of this operator were thoroughly investigated; if λ is a Liouville number, the spectrum of H_λ is a Cantor set [**BS82**]. We note that H_λ is a Schrödinger operator on the one-dimensional lattice. By Fourier transform, it can be realized as an element of the cross product C^*-algebra $R_\lambda \ltimes \mathcal{C}(\mathbb{S}^1)$, where R_λ is the dynamical system generated by an angle-λ rotation on $\mathbb{S}^1$ and $\mathcal{C}(\mathbb{S}^1)$ denotes the algebra of continuous functions on the circle.

Another example was studied by David Kazhdan [**Kaž65**]. Let α and β be two noncommuting rotations in the plane $\mathbb{R}^2$. They generate a group G with a unitary action π on $L^2(\mathbb{R}^2)$. Kazhdan studies the operator $M = \pi(\alpha) + \pi(\alpha^{-1}) + \pi(\beta) + \pi(\beta^{-1})$ and shows that its Fourier transform decomposes as a direct integral of operators acting on functions on the circle. Each of these is an element of $R_\lambda \ltimes \mathcal{B}(\mathbb{S}^1)$, where $\mathcal{B}(\mathbb{S}^1)$ denotes the bounded functions on the circle, and happens to be a Schrödinger operator; spectral properties of these operators are then used to show that the orbits of G in $\mathbb{R}^2$ are uniformly distributed.

E.1.4. Main Results. We produce examples of operators of Hecke type with Cantor set spectrum, but where additionally the representations π involved are quasi-regular. This produces graphs whose Laplace operators have totally discontinuous spectrum, namely the associated Schreier graphs. Our main results read:

THEOREM E.1. (1) *There is a connected 4-regular graph of polynomial growth, which is a Schreier graph of a group of intermediate growth, and whose Laplacian's spectrum is a Cantor set.*

(2) *There is a connected 4-regular graph of polynomial growth, which is a Schreier graph of a group of intermediate growth, and whose Laplacian's spectrum is the union of a Cantor set K and a countable set P of isolated points whose accumulation set is K.*

(3) *There are noncommutative dynamical systems generated by 2 transformations whose spectrum are the same as in the above two points.*

(4) *The above spectra are calculated explicitly. The Cantor set K is of the form $F(J)$ where F is a simple algebraic function and J is the Julia set of a quadratic map $z \mapsto z^2 - \lambda$, where $\lambda = 45/16$ in the first case and $\lambda = 6$ in the second case. J is the set of points of the form*

$$\pm\sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\dots}}}}.$$

To the best of our knowledge, these are the first examples of graphs of constant vertex degree whose spectrum is totally disconnected. There are, however, examples of Schrödinger operators on $\mathbb{Z}$ (or $\mathbb{R}$) with nowhere dense spectrum; they are obtained as Harper operators (as mentioned above) or following a result by Jürgen Moser [**Mos81**].

There are also examples of random walks on non-regular graphs, but with vertex degrees 1 or 3, whose spectrum is the union of a countable set and a Cantor set of null Lebesgue measure [**Mal95**].

Similarly we produce Hecke type operators of quasi-regular representations that have the same spectra as above. These are probably the first examples of quasi-regular representations of virtually torsion-free groups with totally discontinuous spectrum; at least, the representations π_λ of the Heisenberg group are not quasi-regular, but come from the cross-product construction, which is often used to produce interesting examples. In the sequel we produce interesting examples of spectra using purely non-commutative dynamical systems and associated methods.

The graphs mentioned in Theorem E.1 are Schreier graphs of some fractal groups. They are of polynomial growth and have a clear "fractal" appearance; see Figure E.5.2. By a fractal group we mean a group which acts on a regular rooted tree $\mathcal{T}$, such that this action has some self-similar properties; this notion is very much related to that of branch group introduced in [**Gri00**]. The Schreier graphs $\mathcal{S}(G, P, S)$ are defined in E.26; in our examples we take for P the stabilizer $\mathsf{Stab}_G(e)$ of an infinite ray starting at the root of $\mathcal{T}$, i.e. an element of the boundary $\partial\mathcal{T}$.

Let us note that the Schreier graphs we obtained have either 2 ends (and linear growth) or a Cantor set of ends (and superpolynomial growth). This is in accordance with [**Pau99**, Theorem 3.1].

The parabolic subgroups $P = \mathsf{Stab}_G(e)$ have the remarkable property of being *weakly maximal*: $[G : P] = \infty$ but $[G : L] < \infty$ for all $L \gneqq P$. The corresponding quasi-regular representations $\rho_{G/P}$ are irreducible, and $\bigcap_{g\in G} P^g = 1$. We thus have an important family of faithful irreducible representations of G, that deserves further investigation.

The first example of group of fractal type was constructed in [**Gri80a**] as an example of infinite torsion 2-group; it was described as a set of measure-preserving transformations of the interval $[0, 1]$, but can equivalently be described by its action on a rooted tree (the binary expansion of a real in $[0, 1]$ giving a path in the rooted binary tree). Later many new examples of this sort appeared [**GS83b, Gri83, BG99a**]. It then became clear that the study of these groups via their tree action was most fruitful and led to interesting ideas and results; for a survey see [**Gri00**]; and for an introduction to the first example, G, see [**Har00, CMS98**].

Among the five groups we consider, two (Γ and $\overline{\Gamma}$) are virtually torsion-free, and have a totally disconnected spectrum. The existence of such groups lends some hope to the existence of torsion-free fractal groups whose Laplace operator has a totally disconnected spectrum, or at least a gap in the spectrum. Such an example would provide a counterexample to the Kaplansky-Kadison conjecture on idempotents (that implies that the spectrum of the Laplace operator related to the regular representation is connected), and to the Baum-Connes conjecture [**Val89**]. Note that if such a group existed, it would be non-amenable [**HK97**].

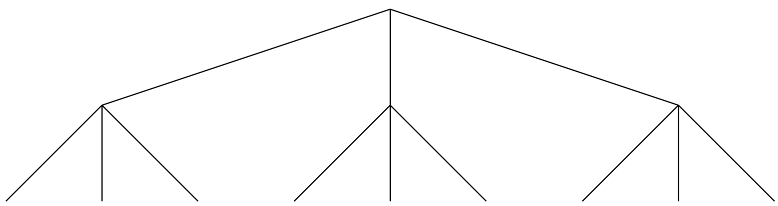
The method used in the computations is the following: we compute explicitly the spectrum of the finite graph $\mathcal{S}(G, P_n, S)$, where P_n is the stabilizer of the rightmost vertex in the nth row of the tree $\mathcal{T}$. Then some arguments, coming from [**Lub95a, GŻ97, MV98**] but adapted to our goals, are used to obtain the spectrum of the infinite graphs from the finite spectra.

E.1.5. Notation. We assume all groups act on the left on sets, and write $g^h = hgh^{-1}$ and $[g,h] = ghg^{-1}h^{-1}$. We also write $\langle S \rangle$ and $\langle S \rangle^G$ for the subgroup and normal subgroup of G generated by S. The regular representation of G in $\ell^2(G)$ is written ρ_G, and the quasi-regular representation of G in $\ell^2(G/H)$ is written $\rho_{G/H}$. The symmetric group on a set S of cardinality n is written $\mathfrak{S}_S$ or $\mathfrak{S}_n$.

E.1.6. Guide to Quick Reading. We present in this paper five computations of spectra related to groups; however, the reader interested solely in examples of graphs with Cantor-set spectrum may wish to skip the group-theoretic discussion. In this case the sections E.4.3 and E.5 should describe the construction in a fairly self-contained manner.

E.2. Groups acting on rooted trees

The groups we shall consider will all be subgroups of the group $\mathsf{Aut}(\mathcal{T})$ of automorphisms of a regular rooted tree $\mathcal{T}$. Let Σ be a finite alphabet. The vertex set of the tree $\mathcal{T}_\Sigma$ is the set of finite sequences over Σ; two sequences are connected by an edge when one can be obtained from the other by right-adjunction of a letter in Σ. The top node is the empty sequence $\emptyset$, and the children of σ are all the σs, for $s \in \Sigma$. We suppose $\Sigma = \mathbb{Z}/d\mathbb{Z}$, with the operation $\overline{s} = s + 1 \mod d$. Let a, called the *rooted automorphism* of $\mathcal{T}_\Sigma$, be the automorphism of $\mathcal{T}_\Sigma$ defined by $a(s\sigma) = \overline{s}\sigma$: it acts nontrivially on the first symbol only, and geometrically is realized as a cyclic permutation of the d subtrees just below the root.



Fix some Σ and let $\mathcal{T} = \mathcal{T}_\Sigma$. For any subgroup $G < \mathsf{Aut}(\mathcal{T})$, let $\mathsf{Stab}_G(\sigma)$ denote the subgroup of G consisting of the automorphisms that fix the sequence σ, and $\mathsf{Stab}_G(n)$ denote the subgroup of G consisting of the automorphisms that fix all sequences of length n:

$$\mathsf{Stab}_G(\sigma) = \{g \in G | \, g\sigma = \sigma\}, \qquad \mathsf{Stab}_G(n) = \bigcap_{\sigma \in \Sigma^n} \mathsf{Stab}_G(\sigma).$$

The $\mathsf{Stab}_G(n)$ are normal subgroups of finite index of G; in particular $\mathsf{Stab}_G(1)$ is of index at most $d!$. Let G_n be the quotient $G/\mathsf{Stab}_G(n)$. If $g \in \mathsf{Aut}(\mathcal{T})$ is an automorphism fixing the sequence σ, we denote by $g_{|\sigma}$ the element of $\mathsf{Aut}(\mathcal{T})$ corresponding to the restriction to sequences starting by σ:

$$\sigma g_{|\sigma}(\tau) = g(\sigma\tau).$$

As the subtree starting from any vertex is isomorphic to the initial tree $\mathcal{T}_\Sigma$, we obtain this way a map

$$\phi : \begin{cases} \mathsf{Stab}_{\mathsf{Aut}(\mathcal{T})}(1) \to \mathsf{Aut}(\mathcal{T})^\Sigma \\ h \mapsto (h_{|0}, \dots, h_{|d-1}) \end{cases} \tag{36}$$

which is an embedding.

DEFINITION E.2. *A subgroup $G < \mathsf{Aut}(\mathcal{T})$ is* level-transitive *if the action of G on Σ^n is transitive for all $n \in \mathbb{N}$. We shall always implicitly make that assumption.*

G is fractal *if for every vertex σ of $\mathcal{T}_\Sigma$ one has $\mathsf{Stab}_G(\sigma)_{|\sigma} \cong G$, where the isomorphism is given by identification of $\mathcal{T}_\Sigma$ with its subtree rooted at σ.*

For a sequence σ and an automorphism $g \in \mathsf{Aut}(\mathcal{T})$, we denote by g^σ the element of $\mathsf{Aut}(\mathcal{T})$ acting as g on the sequences starting by σ, and trivially on the others:

$$g^\sigma(\sigma\tau) = \sigma g(\tau), \qquad g^\sigma(\tau) = \tau \text{ if } \tau \text{ doesn't start by } \sigma.$$

Let $G < \mathsf{Aut}(\mathcal{T})$ be a group acting faithfully, and transitively on each level, on a rooted tree $\mathcal{T}_\Sigma$. The *rigid stabilizer* of σ is $\mathsf{Rist}_G(\sigma) = \{g^\sigma | \, g \in G\} \cap G$. We say G has *infinite rigid stabilizers* if all the $\mathsf{Rist}_G(\sigma)$ are infinite.

DEFINITION E.3.
(1) *G is a* regular branch *group if it has a finite-index subgroup $K < \mathsf{Stab}_G(1)$ such that*

$$K^\Sigma < \phi(K).$$

(2) *A subgroup $G < \mathsf{Aut}(\mathcal{T})$ is a* branch group *if for every $n \geq 1$ there exists a subgroup $L_n < \mathsf{Aut}(\mathcal{T})$ and an embedding*

$$L_n \times \dots \times L_n \hookrightarrow \mathsf{Stab}_G(n),$$

where the direct product is indexed by Σ^n, the injection is given on each factor by $(\ell, \sigma) \mapsto \ell^\sigma$, and the image is normal of finite index in $\mathsf{Stab}_G(n)$.

(3) *G is a* weak branch *group if all of its rigid stabilizers $\mathsf{Rist}_G(\sigma)$ are infinite.*

If G is fractal, one has for all n an embedding $\mathsf{Stab}_G(n) < G^{d^n}$. Note that the definition of a branch group admits an even more general setting — see [**Gri00**]. Four of our examples will be regular branch groups, and the last one will not be a branch, but rather a weak branch group. The following lemma shows that, for fractal groups, 1 implies 2 implies 3 in Definition E.3.

LEMMA E.4. *If G is a fractal, regular branch group, then it is a branch group. If G is a branch group, then it is a weak branch group.*

PROOF. Assume G is a regular branch group on its subgroup K. Define $L_n = K$ for all n. Clearly $\mathsf{Stab}_G(n)$ contains the direct product $L_n^{\Sigma^n}$, and it is of finite index in G^{Σ^n}, so all the more of finite index in $\mathsf{Stab}_G(n)$. The second implication holds because branch groups are infinite, and 'finite index in infinite group' is stronger than 'infinite'. ☐

In the sequel we shall be concerned with subgroups G of $\mathsf{Aut}(\mathcal{T})$ that are finitely generated, fractal, and contain the rooted automorphism a. These groups will be naturally equipped with the restriction of the map ϕ defined in (36), a descending sequence of normal subgroups $H_n = \mathsf{Stab}_G(n)$ and an approximating sequence of finite quotients $G_n = G/H_n$. These quotients can be seen as subgroups of the symmetric group $\mathfrak{S}_{\Sigma^n}$ on Σ^n. More details on all of these groups and their subgroups appear in [**BG99a**].

E.2.1. Dynamical Systems. Assume as above that a group G generated by a set S acts on the d-regular rooted tree $\mathcal{T} = \{0, \dots, d-1\}^*$. Then G acts naturally on the boundary $\partial\mathcal{T} = \{0, \dots, d-1\}^{\mathbb{N}}$, and this action preserves the uniform Bernoulli measure ν on the compact space $\partial\mathcal{T}$. We associate thus a dynamical system $(G, S, \partial\mathcal{T}, \nu)$ to the group G.

This dynamical system is naturally isomorphic to a dynamical system $(G, S, [0,1], m)$, where m is the Lebesgue measure, and G (generated by S) acts on $[0,1]$ by measure-preserving transformations in the following way: let $g \in G$, and $\gamma \in [0,1]$ a d-adic irrational with base-d expansion $0.\gamma_1\gamma_2\dots$. Then $g(\gamma) = 0.\delta_1\delta_2\dots$, where the infinite sequence $(\gamma_1, \gamma_2, \dots)$ is mapped by g to $(\delta_1, \delta_2, \dots)$. This defines the action of G on a subset of full measure of $[0,1]$.

The orbits of G on $\partial\mathcal{T}$ can be made explicit as follows:

Definition E.5. *Two infinite sequences $\sigma, \tau : \mathbb{N} \to \Sigma$ are* confinal *if there is an $N \in \mathbb{N}$ such that $\sigma_n = \tau_n$ for all $n \geq N$.*

Confinality is an equivalence relation, and equivalence classes are called confinality classes.

Proposition E.6. *Let G be a group acting on a regular rooted tree $\mathcal{T}$, and assume that for any generator $g \in G$ and infinite sequence σ, the sequences σ and $g\sigma$ differ only in finitely many places. Then the confinality classes of the action of G on $\partial\mathcal{T}$ are unions of orbits. If moreover $\mathsf{Stab}_G(\sigma)$ contains the rooted automorphism a for all $\sigma \in \mathcal{T}$, the orbits of the action are confinality classes.*

The dynamics of the actions of a group on the boundary of a tree from the point of view of confinality are investigated in more detail in [**NS99**]. We remark that the five example groups—$G, \tilde{G}, \Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$—we shall consider satisfy the conditions of the proposition above.

E.2.2. Growth of Groups and Parabolic Subgroups. We recall some facts about word-growth of groups and sets on which they act.

Definition E.7. *Let G be a group generated by a finite set S, let X be a set upon which G acts transitively, and choose $x \in X$. The* growth *of X is the function $\gamma : \mathbb{N} \to \mathbb{N}$ defined by*

$$\gamma(n) = |\{gx \in X \mid |g| \leq n\}|,$$

where $|g|$ denotes the minimal length of g when written as a word over S. By the growth of *G we mean the growth of the action of G on itself by left-multiplication.*

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there is a constant $C \in \mathbb{N}$ such that $f(n) < Cg(Cn + C) + C$ for all $n \in \mathbb{N}$, and $f \sim g$ if $f \preceq g$ and $g \preceq f$. The equivalence class of the growth of X is independent of the choice of S and of x.

X is of polynomial growth *if $\gamma(n) \preceq n^d$ for some d. It is of* exponential growth *if $\gamma(n) \succeq e^n$. It is of* intermediate growth *in the remaining cases. This trichotomy does not depend on the choice of x.*

Assume now that G is a group acting on the tree Σ^*, and that a subset $S \subset G$ is given.

Definition E.8. *The* portrait *of $g \in G$ with respect to S is a subtree of Σ^*, with inner vertices labeled by $\mathfrak{S}_\Sigma$ and leaf vertices labeled by $S \cup \{1\}$. It is defined recursively as follows: if $g \in S \cup \{1\}$, the portrait of g is the subtree reduced to the root vertex, labeled by g itself. Otherwise, let $\alpha \in \mathfrak{S}_\Sigma$ be the permutation of the top branches of Σ^* such that $g\alpha^{-1} \in \mathsf{Stab}_G(1)$; let $(g_0, \dots, g_{d-1}) = \phi(g\alpha^{-1})$ and let $\mathcal{T}_i$ be the portrait of g_i. Then the portrait of g is the subtree of Σ^* with α labeling the root vertex and subtrees $\mathcal{T}_0, \dots, \mathcal{T}_{d-1}$ connected to the root.*

The depth *of $g \in G$ is the height (length of a maximal path starting at the root vertex) $\partial(g) \in \mathbb{N} \cup \{\infty\}$ of the portrait of g.*

Therefore the depth of g is finite if and only if the portrait of G is finite. Both are finite for all groups we consider in this paper, and the depth may be estimated using the following lemma:

LEMMA E.9. *Assume S generates G and $\phi : g \mapsto (g_1, \dots, g_d)$ defined in (36) has the property that $|g_i| < |g|$ for all i. Then every $g \in G$ has a finite portrait. If moreover there are constants λ, μ with $\mu/(1-\lambda) < 2$ such that $|g_i| \le \lambda|g| + \mu$ for all i, then asymptotically when $|g| \to \infty$*

$$\partial(g) \le \log_{1/\lambda} |g|.$$

PROOF. Consider the 'level' function

$$F(n) = \max_{|g|<n} \partial(g).$$

Then F is increasing, and by assumption $F(2) = 0$, $F(n) \le F(\lambda n + \mu) + 1$. It then follows that

$$F(n) \le F(\lambda n + \mu) + 1 \le \cdots \le F(\lambda^k n + \lambda^{k-1}\mu + \cdots + \lambda\mu + \mu) + k;$$

let us take for k a natural number satisfying $\lambda^k n + \lambda^{k-1}\mu + \cdots + \lambda\mu + \mu \le 2$, for instance

$$k = \left\lceil \log_\lambda \frac{2 - \mu/(1-\lambda)}{n - \mu/(1-\lambda)} \right\rceil,$$

where $\lceil x \rceil$ is the least integer greater than x. The result follows, because then $F(n) \le k$ and $k \sim \log_{1/\lambda} n$. □

SCHOLIUM E.10. *For all groups considered in this paper, we have $|g_i| \le \frac{1}{2}|g| + \frac{1}{2}$ for their natural generating systems, as can be checked on the tables describing ϕ. As a consequence, they all satisfy $\partial(g) \precsim \log_2 |g|$.*

DEFINITION E.11. *Let $\mathcal{T} = \Sigma^*$ be a rooted tree. A* ray *e in $\mathcal{T}$ is an infinite geodesic starting at the root of $\mathcal{T}$, or equivalently an element of $\partial\mathcal{T} = \Sigma^{\mathbb{N}}$.*

Let $G < \mathsf{Aut}(\mathcal{T})$ and e be a ray. The associated parabolic subgroup *is $\mathsf{Stab}_G(e) = \bigcap_{n\ge 0} \mathsf{Stab}_G(e_n)$, where e_n is the length-n prefix of e.*

Assume that G satisfies the conditions of Lemma E.9. Then we have the

PROPOSITION E.12. *Let $G < \mathsf{Aut}(\mathcal{T})$ satisfy the conditions of Proposition E.6 and Lemma E.9 (for the constant λ), and let P be a parabolic subgroup. Then G/P, as a G-set, is of polynomial growth of degree at most $\log_{1/\lambda}(d)$. If moreover G is level-transitive, then G/P's asymptotical growth is polynomial of degree $\log_{1/\lambda}(d)$.*

PROOF. Suppose that $P = \mathsf{Stab}_G(e)$. Then G/P can be identified with the G-orbit of e, and, by Proposition E.6, with the set of all infinite sequences over Σ that eventually coincide with e. If $\partial(g) < k$, it sends the infinite sequence e to one of the d^k sequences in $\Sigma^k e_k \dots$; thus the image of e under the set of elements of depth at most k is of cardinality bounded by d^k. The image of e under the set of elements of length at most n is then asymptotically bounded by $d^{\log_{1/\lambda}(n)} = n^{\log_{1/\lambda}(d)}$, by Lemma E.9. □

We now recall some facts on commensurators:

DEFINITION E.13. *The* commensurator *of a subgroup H of G is*

$$\mathrm{comm}_G(H) = \{g \in G |\, H \cap H^g \text{ is of finite index in } H \text{ and } H^g\}.$$

Equivalently, letting H act on the left on the cosets $\{gH\}$,

$$\mathrm{comm}_G(H) = \{g \in G |\, H \cdot (gH) \text{ and } H \cdot (g^{-1}H) \text{ are finite orbits}\}.$$

PROPOSITION E.14 ([**BG99a**]). *Let G be a weak branch group and let P be a parabolic subgroup. Then $\mathrm{comm}_G(P) = P$.*

THEOREM E.15 (Mackey [**Mac76, BH97**]). *Let $P < G$ be any subgroup inclusion. Then the quasi-regular representation $\rho_{G/P}$ is irreducible if and only if $\mathrm{comm}_G(P) = P$.*

Therefore, for all weak branch groups $\rho_{G/P}$ is irreducible.

The following lemma is well known:

LEMMA E.16. *Let $H < G$ be a subgroup of finite index. Then the orbits of H on G/H, the double cosets $HgH < G$ and the irreducible components of the G-space $\ell^2(G/H)$ are all in bijection.*

Therefore, the orbits of $P_n = P \cdot \mathsf{Stab}_G(n)$ on Σ^n are in bijection with the decomposition of ρ_{G/P_n} in irreducible subrepresentations.

E.2.3. Groups and Finite Automata. There are various uses of automata in group theory, most notably as *word acceptors*, where the automata recognize some words as group elements and perform operations on these words [**ECH**$^+$**92**]; and as *transducers* or *sequential machines* (see [**Eil74**, Chapter XI] or [**GC71**]), where the automata themselves are the elements of the group, and are distinguished by the transformation they perform on their input. The former use gives rise to the theory of *automatic groups*; we propose to call the latter *automata groups*. The automata they are built with are called *Mealy machines* or *Moore machines* (see [**Glu61**] or [**Bra84**, page 109]).

We present a restricted definition of *finite transducers*. In the standard terminology, they would be called *invertible transducers*.

DEFINITION E.17. *Let Σ be a finite alphabet. A* finite transducer *on Σ is a finite directed graph $\mathfrak{G} = (V, E)$, a labeling $\lambda : E \to \Sigma$ of the edges such that for each vertex $v \in V$ the restriction of λ is a bijection between $\{e \in E |\, \alpha(e) = v\}$ and Σ, and a labeling $\tau : V \to \mathfrak{S}_\Sigma$ of the nodes (called* states*) by the symmetric group on Σ.*

An initial transducer *$\mathfrak{G}_q$ is a finite transducer $\mathfrak{G}$ with a distinguished initial state $q \in V$.*

Let $\mathfrak{G}$ be a finite transducer, and $\{\mathfrak{G}_q\}_{q\in V}$ be the set of its initial transducers. Each $\mathfrak{G}_q$ defines an automorphism $\overline{\mathfrak{G}_q}$ of the rooted tree $\mathcal{T}_\Sigma$ as follows: let $\sigma = \sigma_0 \dots \sigma_n$ be a vertex of $\mathcal{T}$. Let e be the edge of $\mathfrak{G}_q$ labeled σ_0. Define recursively

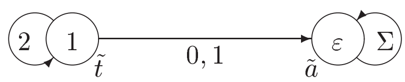
$$\overline{\mathfrak{G}_q}(\sigma_0 \dots \sigma_n) = \tau(q)(\sigma_0)\overline{\mathfrak{G}_{\omega(e)}}(\sigma_1 \dots \sigma_n).$$

We shall call two initial transducers $\mathfrak{G}_q$ and $\mathfrak{G}'_{q'}$ *equivalent* if their actions $\overline{\mathfrak{G}_q}$ and $\overline{\mathfrak{G}'_{q'}}$ on Σ^* are the same. Every initial transducer is equivalent to a unique transducer that is minimal with respect to its number of nodes [**Eil74**, Chapter XII, Theorem 4.1].

Define now $G(\mathfrak{G})$ as the group generated by the $\overline{\mathfrak{G}_q}$, where q ranges over the set of states of $\mathfrak{G}$. We call such a group an *automata group*. The following fact is well known, and dates back to Jiří Hořejš in the early 60's [**Hoř63**]:

PROPOSITION E.18. *Let $\mathfrak{G}_q$ and $\mathfrak{G}'_{q'}$ be finite initial transducers on the same alphabet Σ. Then $\overline{\mathfrak{G}_q}^{-1}$ and $\overline{\mathfrak{G}_q} \circ \overline{\mathfrak{G}'_{q'}}$ can be represented as finite initial transducers.*

In general different transducers can generate isomorphic groups. For instance, consider the three-vertex transducer in the middle of Figure E.2. The group it generates is isomorphic (and even conjugate in $\mathsf{Aut}(\mathcal{T})$) to the one generated by the following two-state transducer $\mathfrak{G}$ on $\Sigma = \{0, 1, 2\}$, because the elements $\tilde{t}, \tilde{a}$ satisfy the same recursions as t, a (see Subsection E.3.3):



Here ε is the cycle $(0, 1, 2) \in \mathfrak{S}_3$. The actions of $\mathfrak{G}_{\tilde{t}}$ and $\mathfrak{G}_{\tilde{a}}$ are as follows:

$$\mathfrak{G}_{\tilde{t}}(2 \dots 2\sigma_m \dots \sigma_n) = 2 \dots 2\varepsilon(\sigma_m) \dots \varepsilon(\sigma_n) \qquad \text{when } \sigma_m \neq 2,$$
$$\mathfrak{G}_{\tilde{a}}(\sigma_0 \dots \sigma_n) = \varepsilon(\sigma_0) \dots \varepsilon(\sigma_n).$$

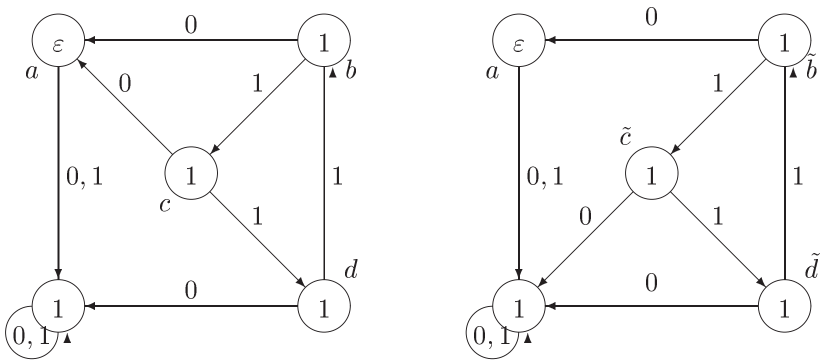


Figure E.1: The Finite Transducers for G and $\tilde{G}$

E.2.4. The Group G. We give here some basic facts about the first of our examples, the group G [**Gri80a, Gri84**]. We take $\Sigma = \{0, 1\}$. Recall a is the automorphism permuting the top two branches of $\mathcal{T}_2$. Let recursively b be the automorphism acting as a on the right branch and c on the left, c be the automorphism acting as a on the right branch and d on the left, and d be the automorphism acting as 1 on the right branch and b on the left. In formulæ,

$$\begin{aligned} b(0x\sigma) &= 0\overline{x}\sigma, & b(1\sigma) &= 1c(\sigma), \\ c(0x\sigma) &= 0\overline{x}\sigma, & c(1\sigma) &= 1d(\sigma), \\ d(0x\sigma) &= 0x\sigma, & d(1\sigma) &= 1b(\sigma). \end{aligned}$$

G is the group generated by $\{a, b, c, d\}$. It is readily checked that these generators are of order 2 and that $\{1, b, c, d\}$ constitutes a Klein group; one of the generators $\{b, c, d\}$ can thus be omitted.

G was originally defined in [**Gri80a**] as the following dynamical system acting on the interval $[0, 1]$ from which rational dyadic points are removed:

$$a(z) = \begin{cases} z + \frac{1}{2} & \text{if } z < \frac{1}{2} \\ z - \frac{1}{2} & \text{if } z \geq \frac{1}{2}, \end{cases}$$

$$\begin{array}{rl} b(z) = & \underset{0}{|}\quad a \quad \underset{\frac{1}{2}}{|}\quad a \quad \underset{\frac{3}{4}}{|}\; 1 \;\underset{\frac{7}{8}}{|}\; a.. \;\underset{1}{|} \\ c(z) = & |\quad a \quad |\quad 1 \quad |\; a \;|\; a.. \;| \\ d(z) = & |\quad 1 \quad |\quad a \quad |\; a \;|\; 1.. \;| \end{array}$$

Here the intervals represent $[0, 1]$, with either a or 1 (the identity transformation) acting on the described subintervals in a similar way as a or 1 act on $[0, 1]$. Finally, G is also an automata group, see the left graph in Figure E.1 (the trivial and non-trivial elements of $\mathfrak{S}_2$ are represented as 1 and ϵ and are used to label states).

Recall the map ϕ defined in (36); it restricts to an embedding $\phi : H \to G \times G$ given by

$$\phi : \begin{cases} b \to (a, c), & b^a \to (c, a) \\ c \to (a, d), & c^a \to (d, a) \\ d \to (1, b), & d^a \to (b, 1), \end{cases} \tag{37}$$

where $H = \mathsf{Stab}_G(1) = \langle b, c, d \rangle^G$ is an index-2 subgroup. Consider also $K = \langle (ab)^2 \rangle^G$. Let e be the infinite sequence 0^∞; set $P_n = \mathsf{Stab}_G(0^n)$ and $P = \mathsf{Stab}_G(e) = \cap_{n \geq 0} P_n$ as above. Clearly P_n has index 2^n in G, as G acts transitively on Σ^n, and P has infinite index.

We note the following facts about G: it

- is an infinite torsion 2-group;
- is of intermediate growth, and therefore amenable (see Definition E.28);
- is fractal, and regular branch on its subgroup K;
- is just infinite;
- has a recursive presentation with infinitely many relators; these relators are obtained as iterates of a substitution on a finite set of words [**Lys85**];
- is residually finite, and more precisely has a natural sequence of finite approximating quotients $G_n = G/\operatorname{\mathsf{Stab}}_G(n)$, of order $2^{5\cdot 2^{n-3}+2}$ for $n \geq 3$ (and order 2^{2^n-1} for $n \leq 3$);
- has a faithful action on the set G/P, of linear growth by Proposition E.12;
- has a faithful action an $\partial\mathcal{T}$ whose orbits are confinality classes.

The decomposition of ρ_{G/P_n} in irreducibles is given by the following lemma, combined with Lemma E.16:

Lemma E.19 ([**BG99a**]). *P_n has $n+1$ orbits in Σ^n; they are 0^n and the $0^i1\Sigma^{n-1-i}$ for $0 \leq i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $0^i\Sigma^*$ for all $i \in \mathbb{N}$.*

E.2.5. The Group $\tilde{G}$. We describe briefly another fractal group, acting on the same tree $\mathcal{T}_2$ as G. More details appear in [**BG99a**]. We denote again by a the automorphism permuting the top two branches, and let recursively $\tilde{b}$ be the automorphism acting as a on the right branch and $\tilde{c}$ on the left, $\tilde{c}$ be the automorphism acting as 1 on the right branch and $\tilde{d}$ on the left, and $\tilde{d}$ be the automorphism acting as 1 on the right branch and $\tilde{b}$ on the left. In formulæ,

$$\begin{aligned}
\tilde{b}(0x\sigma) &= 0\overline{x}\sigma, & \tilde{b}(1\sigma) &= 1\tilde{c}(\sigma),\\
\tilde{c}(0\sigma) &= 0\sigma, & \tilde{c}(1\sigma) &= 1\tilde{d}(\sigma),\\
\tilde{d}(0\sigma) &= 0\sigma, & \tilde{d}(1\sigma) &= 1\tilde{b}(\sigma).
\end{aligned}$$

Then $\tilde{G}$ is the group generated by $\{a, \tilde{b}, \tilde{c}, \tilde{d}\}$. Clearly all these generators are of order 2, and $\{\tilde{b}, \tilde{c}, \tilde{d}\}$ is elementary abelian of order 8. It can be defined using the second automaton in Figure E.1, or as the dynamical system

$$a(z) = \begin{cases} z + \frac{1}{2} & \text{if } z < \frac{1}{2} \\ z - \frac{1}{2} & \text{if } z \geq \frac{1}{2}, \end{cases}$$

$$\begin{aligned}
\tilde{b}(z) &= \begin{array}{|c|c|c|c|} \hline \quad\quad a \quad\quad & \quad 1 \quad & 1 & a.. \\ \hline \end{array}\\
\tilde{c}(z) &= \begin{array}{|c|c|c|c|} \hline \quad\quad 1 \quad\quad & \quad 1 \quad & a & 1.. \\ \hline \end{array}\\
\tilde{d}(z) &= \begin{array}{|c|c|c|c|} \hline \quad\quad 1 \quad\quad & \quad a \quad & 1 & 1.. \\ \hline \end{array}
\end{aligned}$$

Recall the map ϕ defined in (36); it restricts to an embedding $\phi : \tilde{H} \to \tilde{G} \times \tilde{G}$ given by

$$\phi : \begin{cases} \tilde{b} \to (a, \tilde{c}), & \tilde{b}^a \to (\tilde{c}, a) \\ \tilde{c} \to (1, \tilde{d}), & \tilde{c}^a \to (d, 1) \\ \tilde{d} \to (1, \tilde{b}), & \tilde{d}^a \to (\tilde{b}, 1), \end{cases}$$

where $\tilde{H} = \operatorname{\mathsf{Stab}}_{\tilde{G}}(1) = \langle \tilde{b}, \tilde{c}, \tilde{d}\rangle^{\tilde{G}}$ is an index-2 subgroup. Consider also $\tilde{K} = \langle (a\tilde{b})^2, (a\tilde{d})^2\rangle^{\tilde{G}}$. Let again e be the infinite sequence 0^∞; set $\tilde{P}_n = \operatorname{\mathsf{Stab}}_{\tilde{G}}(0^n)$ of index 2^n, and $\tilde{P} = \operatorname{\mathsf{Stab}}_{\tilde{G}}(e) = \cap_{n\geq 0}\tilde{P}_n$ of infinite index.

We note the following facts about G, proved in [**BG99a**]: it

- is an infinite group containing $G = \langle a, \tilde{b}\tilde{c}, \tilde{c}\tilde{d}, \tilde{d}\tilde{b}\rangle$ as an infinite-index subgroup, and has 2-torsion elements as well as infinite-order elements;

- is of intermediate growth, and therefore is amenable;
- is fractal, and regular branch on its subgroup $\tilde{K}$;
- is just infinite;
- has a recursive presentation with infinitely many relators; these relators are obtained as iterates of a substitution on a finite set of words. All of $\tilde{G}$'s relators have even length with respect to the generating set $\{a, \tilde{b}, \tilde{c}, \tilde{d}\}$, so its Cayley graph is bipartite;
- is residually finite, and more precisely has a natural sequence of finite approximating quotients $\tilde{G}_n = \tilde{G}/\operatorname{\mathsf{Stab}}_{\tilde{G}}(n)$, of order $2^{13\cdot 2^{n-4}+2}$ for $n \geq 4$ (and order 2^{2^n-1} for $n \leq 4$).
- has a faithful action on the set $\tilde{G}/\tilde{P}$, of linear growth by Proposition E.12;
- has a faithful action an $\partial\mathcal{T}$ whose orbits are confinality classes.

The decomposition of $\rho_{\tilde{G}/\tilde{P}_n}$ in irreducibles is given by the following lemma, combined with Lemma E.16:

Lemma E.20. *$\tilde{P}_n$ has $n+1$ orbits in Σ^n; they are 0^n and the $0^i 1 \Sigma^{n-1-i}$ for $0 \leq i < n$. The orbits of $\tilde{P}$ in $\mathcal{T}_\Sigma$ are the $0^i\Sigma^*$ for all $i \in \mathbb{N}$.*

E.2.6. *GGS* groups. We next study three examples of a family of groups called *GGS* groups (the terminology was introduced by Gilbert Baumslag [**Bau93**] and refers to Rostislav Grigorchuk, Narain Gupta and Said Sidki). Let p be a prime number. Denote a the automorphism of $\mathcal{T}_p$ permuting cyclically the top p branches. Let $\epsilon = (\epsilon_0, \dots, \epsilon_{p-2}) \in (\mathbb{Z}/p)^{p-1}$. Define recursively the automorphism t_ϵ of $\mathcal{T}_p$ by

$$t(xy\sigma) = x(y + \epsilon_x)\sigma \text{ if } 0 \leq x \leq p-2, \qquad t((p-1)\sigma) = (p-1)t(\sigma).$$

Then G_ϵ is the subgroup of $\operatorname{\mathsf{Aut}}(\mathcal{T}_p)$ generated by $\{a, t\}$.

The following results have their roots in [**Gri80a, GS83b**] and are known as part of folklore; for a proof see [**Gri00**]:

Theorem E.21. *G_ϵ is an infinite group if an only if $\epsilon \neq (0, \dots, 0)$. It is a torsion group if and only if $\sum \epsilon_i = 0$.*

Theorem E.22 ([**Bar00a**]). *For all ϵ, the group G_ϵ is of subexponential growth, and therefore is amenable.*

Let $G = G_\epsilon$ be as above. Let $e = (p-1)^\infty$ be the rightmost path in $\mathcal{T}_p$ and let $P = \operatorname{\mathsf{Stab}}_G(e)$. Proposition E.12 applies, so G/P has polynomial growth of degree at most $\log_2(p)$.

E.2.7. The Group Γ. As always denote by a the rooted automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let s be the automorphism of $\mathcal{T}_3$ defined recursively by

$$s(0x\sigma) = 0\overline{x}\sigma, \qquad s(1x\sigma) = 1x\sigma, \qquad s(2\sigma) = 2s(\sigma).$$

Then Γ is the subgroup of $\operatorname{\mathsf{Aut}}(\mathcal{T}_3)$ generated by $\{a, s\}$; its growth was studied by Jacek Fabrykowski and Narain Gupta [**FG91**]. It can also be defined using automata, as in Figure E.2, or as the dynamical system

$$a(z) = \begin{cases} z + \frac{1}{3} & \text{if } z < \frac{2}{3} \\ z - \frac{2}{3} & \text{if } z \geq \frac{2}{3}, \end{cases}$$

$$s(z) = \begin{array}{c|c|c|c|c|c} & a & 1 & a & 1 & a.. \\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{7}{9} & \frac{8}{9} & 1 \end{array}$$

Set $H = \operatorname{\mathsf{Stab}}_\Gamma(1) = \langle s \rangle^\Gamma$; then $\phi : H \to \Gamma \times \Gamma \times \Gamma$ can be expressed as

$$\phi : \left\{ s \to (a, 1, s), \quad s^a \to (s, a, 1), \quad s^{a^2} \to (1, s, a). \right. \tag{38}$$

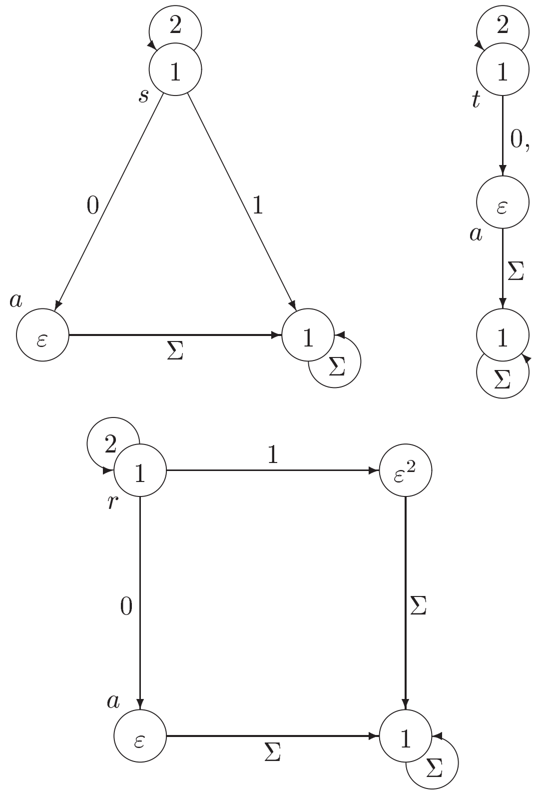


Figure E.2: The Automata for Γ, $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$

Define the elements $x = as$, $y = sa$ of Γ, and let K be the subgroup of Γ generated by x and y. Then K is normal in Γ, because $y^s = x^{-1}y^{-1}$, $y^{a^{-1}} = y^{-1}x^{-1}$, $y^{s^{-1}} = y^a = a$, and similar relations hold for conjugates of x. Moreover K is of index 3 in Γ, with transversal $\langle a\rangle$. Let L be the subgroup of K generated by $[K, K]$ and cubes in K.

Theorem E.23 ([**BG99a**]). *Γ is a regular branch, fractal group. The subgroup K of Γ is torsion-free; thus Γ is virtually torsion-free. The finite quotients $\Gamma_n = \Gamma/H_n$ of Γ have order $3^{3^{n-1}+1}$ for $n \geq 2$, and 3 for $n = 1$.*

E.2.8. The Group $\overline{\Gamma}$. Let a be as in the previous subsection, and let t be the automorphism of $\mathcal{T}_3$ defined recursively by

$$t(0x\sigma) = 0\overline{x}\sigma, \qquad t(1x\sigma) = 1\overline{x}\sigma, \qquad t(2\sigma) = 2t(\sigma).$$

Then $\overline{\Gamma}$ is the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, t\}$. The associated dynamical system is

$$a(z) = \begin{cases} z + \frac{1}{3} & \text{if } z < \frac{2}{3} \\ z - \frac{2}{3} & \text{if } z \geq \frac{2}{3}, \end{cases}$$

$$t(z) = \underline{|\quad a \quad|\quad a \quad| \; a \; | \; a \; | a..|}.$$

Set $H = \mathsf{Stab}_{\overline{\Gamma}}(1) = \langle t\rangle^{\overline{\Gamma}}$; then $\phi : H \to \overline{\Gamma} \times \overline{\Gamma} \times \overline{\Gamma}$ can be expressed as

$$\phi : \Big\{ t \to (a, a, t), \quad t^a \to (t, a, a), \quad t^{a^2} \to (a, t, a).$$

Define the elements $x = ta^{-1}$, $y = a^{-1}t$ of $\overline{\Gamma}$, and let $\overline{K}$ be the subgroup of $\overline{\Gamma}$ generated by x and y. Then $\overline{K}$ is normal in $\overline{\Gamma}$, because $x^t = y^{-1}x^{-1}$, $x^a = x^{-1}y^{-1}$, $x^{t^{-1}} = x^{a^{-1}} = y$, and similar relations hold for conjugates of y. Moreover $\overline{K}$ is of index 3 in $\overline{\Gamma}$, with transversal $\langle a\rangle$.

Theorem E.24 ([**BG99a**]). *$\overline{\Gamma}$ is a fractal group and is weak branch, but not branch. The subgroup $\overline{K}$ of $\overline{\Gamma}$ is torsion-free; thus $\overline{\Gamma}$ is virtually torsion-free. The finite quotients $\overline{\Gamma}_n = \overline{\Gamma}/H_n$ of $\overline{\Gamma}$ have order $3^{\frac{1}{4}(3^n+2n+3)}$ for $n \geq 2$, and $3^{\frac{1}{2}(3^n-1)}$ for $n \leq 2$.*

E.2.9. The Group $\overline{\overline{\Gamma}}$. Let again a denote the automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let now r be the automorphism of $\mathcal{T}_3$ defined recursively by

$$r(0x\sigma) = 0\overline{x}\sigma, \qquad r(1x\sigma) = 1\overline{\overline{x}}\sigma, \qquad r(2\sigma) = 2r(\sigma).$$

Then $\overline{\overline{\Gamma}}$ is the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, r\}$; it was studied by Gupta and Sidki [**GS83a, GS83b, Sid87a, Sid87b**]. The associated dynamical system is

$$a(z) = \begin{cases} z + \frac{1}{3} & \text{if } z < \frac{2}{3} \\ z - \frac{2}{3} & \text{if } z \geq \frac{2}{3}, \end{cases}$$

$$r(z) = \underline{|\quad a \quad|\quad a^2 \quad|\ a\ |\ a^2\ |\ a..|}.$$

Set $H = \mathsf{Stab}_{\overline{\overline{\Gamma}}}(1) = \langle r\rangle^{\overline{\overline{\Gamma}}}$; then $\phi : H \to \overline{\overline{\Gamma}} \times \overline{\overline{\Gamma}} \times \overline{\overline{\Gamma}}$ con be expressed as

$$\phi : \Big\{ r \to (a, a^2, r), \quad r^a \to (r, a, a^2), \quad r^{a^2} \to (a^2, r, a).$$

Theorem E.25 ([**BG99a**]). *$\overline{\overline{\Gamma}}$ is a just infinite torsion 3-group. It is branch and fractal. The finite quotients $\overline{\overline{\Gamma}}_n = \overline{\overline{\Gamma}}/H_n$ of Γ have order $3^{2\cdot 3^{n-2}+1}$ for $n \geq 2$, and 3 for $n = 1$.*

E.3. Unitary Representations and Hecke type Operators

The five groups introduced in the previous section share the property of acting faithfully on a regular rooted tree; natural representations arise from this fact. In this chapter we suppose G is any group acting level-transitively on a regular tree.

We defined in Subsection E.2.1 the boundary $\partial\mathcal{T}$ of the tree on which G acts. Since G preserves the uniform measure on this boundary, we have a unitary representation π of G in $L^2(\partial\mathcal{T}, \nu)$, or equivalently in $L^2([0,1], m)$. Let $\mathcal{H}_n$ be the subspace of $L^2(\partial\mathcal{T}, \nu)$ spanned by the characteristic functions χ_σ of the rays e starting by σ, for all $\sigma \in \Sigma^n$. It is of dimension d^n, and can equivalently be seen as spanned by the characteristic functions in $L^2([0,1], m)$ of intervals of the form $[(i-1)d^{-n}, id^{-n}]$, $1 \leq i \leq d^n$. These $\mathcal{H}_n$ are invariant subspaces, and afford representations $\pi_n = \pi_{|\mathcal{H}_n}$. As clearly π_{n-1} is a subrepresentation of π_n, we set $\pi_n^\perp = \pi_n \ominus \pi_{n-1}$, so that $\pi = \oplus_{n=0}^\infty \pi_n^\perp$.

Denote by $\rho_{G/H}$ the quasi-regular representation of G in $\ell^2(G/H)$ and by ρ_{G/H_n} the finite-dimensional representations of G in $\ell^2(G/H_n)$. Since G is level-transitive, the representations π_n and ρ_{G/H_n} are unitary equivalent.

Definition E.26. *Let G be a group generated by a set S and H a subgroup of G. The* Schreier graph *$\mathcal{S}(G, H, S)$ of G/H is the directed graph on the edge set G/H, with for every $s \in S$ and every $gH \in G/H$ an edge from gH to sgH. The base point of $\mathcal{S}(G, H, S)$ is the coset H.*

Note that $\mathcal{S}(G, 1, S)$ is the Cayley graph of G relative to S. It may happen that $\mathcal{S}(G, P, S)$ have loops and multiple edges even if S is disjoint from H. Schreier graphs are $|S|$-regular graphs, and any degree-regular graph $\mathfrak{G}$ containing a 1-factor (i.e. a regular subgraph of degree 1; there is always one if $\mathfrak{G}$ has even degree) is a Schreier graph [**Lub95a**, Theorem 5.4].

DEFINITION E.27. *Let G be a group generated by a finite symmetric set S. The* spectrum $\operatorname{spec}(\tau)$ *of a representation $\tau : G \to \mathcal{U}(\mathcal{H})$ with respect to the given set of generators is the spectrum of $\Delta_\tau = \sum_{s\in S} \tau(s)$ seen as an bounded operator on $\mathcal{H}$.*

As the vertices of $\mathcal{S}(G, H, S)$ coincide with the set G/H, it is easy to see that $\Delta_\tau/|S|-1$ is unitary equivalent to the Laplacian operator on $\mathcal{S}(G, H, S)$, and therefore has same spectrum.

DEFINITION E.28. *Let G be a group acting on a set X. This action is* amenable *in the sense of von Neumann* [**vN29**] *if there exists a finitely additive measure μ on X, invariant under the action of G, with $\mu(X) = 1$.*

A group G is amenable *if its action on itself by left-multiplication is amenable.*

Amenability can be tested using the following criterion, due to Følner for the regular action [**Føl57**] (see also [**HGC99**] and the literature cited there):

THEOREM E.29. *Assume the group G acts on a discrete set X. Then the action is amenable if and only for every if for every $\lambda > 0$ and every $g \in G$ there exists a finite subset $F \subset X$ such that $|F \triangle gF| < \lambda|F|$, where $\triangle$ denotes symmetric difference and $|\cdot|$ cardinality.*

Using this criterion it is easy to see that any G-space X of subexponential growth is amenable. In particular the G-spaces G/P are of polynomial growth when the conditions of Proposition E.12 are fulfilled, and therefore are amenable. The following result belongs to the common lore, and we don't know of a reference to its proof. Our attention was drawn to it by Marc Burger and Alain Valette:

PROPOSITION E.30. *Let $H < G$ be any subgroup. Then the quasi-regular representation $\rho_{G/H}$ is weakly contained in ρ_G if and only if H is amenable.*

PROOF. If H is amenable, then the trivial one-dimensional representation 1_H of H is weakly contained in ρ_H. Inducing up, $\rho_{G/H} = \operatorname{Ind}_H^G 1_H$ is weakly contained in $\rho_G = \operatorname{Ind}_H^G(\rho_H)$.

Conversely, if $\rho_{G/H}$ is weakly contained in ρ_G, we obtain by restricting to H that $\rho_{G/H|H}$ is weakly contained in $\rho_{G|H}$. Now 1_H is a subrepresentation of $\rho_{G/H|H}$, because the Dirac mass at H is H-fixed; and $\rho_{G|H} = [G : H]\rho_H$ by Frobenius reciprocity. It follows that 1_H is weakly contained in $[G : H]\rho_H$, and therefore that H is amenable. □

The following statements allows one to compare spectra of diverse representations.

THEOREM E.31. *Let G be a group acting on a regular rooted tree, and let π, π_n and $\pi_n^\perp$ be as above.*

(1) *If G is weak branch, then $\rho_{G/P}$ is an irreducible representation of infinite dimension.*
(2) *π is a reducible representation of infinite dimension whose irreducible components are precisely those of the $\pi_n^\perp$ (and thus are all finite-dimensional). Moreover*
$$\operatorname{spec}(\pi) = \overline{\bigcup_{n\geq 0} \operatorname{spec}(\pi_n)} = \overline{\bigcup_{n\geq 0} \operatorname{spec}(\pi_n^\perp)}.$$
(3) *The spectrum of $\rho_{G/P}$ is contained in $\overline{\cup_{n\geq 0} \operatorname{spec}(\rho_{G/P_n})} = \overline{\cup_{n\geq 0} \operatorname{spec}(\pi_n)}$, and thus is contained in the spectrum of π. If moreover either P or G/P are amenable, these spectra coincide, and if P is amenable, they are contained in the spectrum of ρ_G:*
$$\operatorname{spec}(\rho_{G/P}) = \operatorname{spec}(\pi) = \overline{\bigcup_{n\geq 0} \operatorname{spec}(\pi_n)} \subseteq \operatorname{spec}(\rho_G).$$
(4) *Δ_π has a pure-point spectrum, and its spectral radius $r(\Delta_\pi) = s \in \mathbb{R}$ is an eigenvalue, while the spectral radius $r(\Delta_{\rho_{G/P}})$ is not an eigenvalue of $\Delta_{\rho_{G/P}}$. Thus $\Delta_{\rho_{G/P}}$ and Δ_π are different operators having the same spectrum.*

PROOF. The first statement follows from Mackey's theorem E.15 and Proposition E.14. The second holds because π splits as the direct sum of the $\pi_n^\perp$'s, and is weakly equivalent to the direct sum of the π_n's.

It was mentioned that ρ_{G/P_n} and π_n are equivalent: they are both finite-dimensional and act on G-equivalent sets, namely G/H_n and Σ^n. The third statement then follows from Proposition E.30 and Propositions E.33 and E.34 below.

It is obvious that s is an eigenvalue of Δ_π with constant eigenfunction. Now the fourth follows from Proposition 5 in [**GŻ97**]. □

Since all of our example groups are amenable, the spectra computed in the next section are included in $\operatorname{spec}(\rho_G)$. Moreover, as $\tilde{G}$ has a bipartite Cayley graph, $\operatorname{spec}(\rho_{\tilde{G}})$ is symmetrical about 1 and contains $[0,4]$ (as we shall show in Section E.4.2), so is $[-4,4]$.

We finish this subsection by turning to a question of Mark Kać [**Kać96**]: "Can one hear the shape of a drum?" This question was answered in the negative in [**GWW92**], and we here answer by the negative to a related question: "Can one hear a representation?" Indeed $\rho_{G/P}$ and π have same spectrum (i.e. cannot be distinguished by hearing), but are not equivalent. Furthermore, if G is a branched group, there are uncountably many nonequivalent representations within $\{\rho_{G/\mathsf{Stab}_G(e)} | \, e \in \partial\mathcal{T}\}$, as is shown in [**BG99a**].

The same question may be asked for graphs: "are there two non-isomorphic graphs with same spectrum?" There are finite examples, obtained through the notion of *Sunada pair* [**Lub95a**]. Cédric Béguin, Alain Valette and Andrzej Żuk produced the following example in [**BVŻ97**]: let Γ be the integer Heisenberg group (free 2-step nilpotent on 2 generators x, y). Then $\Delta = x + x^{-1} + y + y^{-1}$ has spectrum $[-2,2]$, which is also the spectrum of $\mathbb{Z}^2$ for an independent generating set. As a consequence, their Cayley graphs have same spectrum, but are not quasi-isometric (they do not have the same growth).

Using the result of Nigel Higson and Gennadi Kasparov [**HK97**] (giving a partial positive answer to the Baum-Connes conjecture), we may infer the following

PROPOSITION E.32. *Let Γ be a torsion-free amenable group with finite generating set $S = S^{-1}$ such that there is a map $\phi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ with $\phi(S) = \{1\}$. Then*

$$\operatorname{spec}(\sum_{s \in S} \rho(s)) = [-|S|, |S|].$$

In particular, there are countably many non-quasi-isometric graphs with the same spectrum, including the graphs of $\mathbb{Z}^d$, of free nilpotent groups and of suitable torsion-free groups of intermediate growth (for the first examples, see [**Gri85**]).

PROOF. Since Γ is amenable, $|S|$ is in its spectrum. By the existence of ϕ, the Cayley graph of Γ with respect to S is two-colourable (i.e. bipartite), so its spectrum is symmetrical, and therefore contains $-|S|$. By the Baum-Connes conjecture, proved for this case by Higson and Kasparov, the C^*-algebra $C_r^*(\rho(\Gamma))$ contains no idempotents. It follows by functional integration that ρ's spectrum is connected, so is $[-|S|, |S|]$ as claimed. □

E.3.1. Approximations of Operators and Spectra. We now prove the claimed inclusions of spectra, in part relying on C^*-algebraic results.

PROPOSITION E.33. *Let $\{H_n\}_{n \geq 0}$ be a descending family of finite-index subgroups of G, and set $H = \bigcap_{n \geq 0} H_n$. Let τ and τ_n be the quasi-regular representations of G on G/H and G/H_n respectively. Then*

$$\operatorname{spec} \tau \subseteq \overline{\bigcup_{n \geq 0} \operatorname{spec} \tau_n}.$$

PROOF. Let $\mathcal{S}$ and $\mathcal{S}_n$ be the Schreier graphs $\mathcal{S}(G, H, S)$ and $\mathcal{S}(G, H_n, S)$ respectively, and mark in them the vertices H and H_n. Then $\mathcal{S}_n \underset{n\to\infty}{\Longrightarrow} \mathcal{S}$ in the sense of [**GŻ97**], that is, coincide with $\mathcal{S}$ in ever increasing balls centered at H_n; therefore, for their associated spectral measures, $\sigma_n(\lambda) \underset{n\to\infty}{\Longrightarrow} \sigma(\lambda)$, in the sense of weak convergence. Therefore the support of $d\sigma$ is contained in the closure of the union of the supports of $d\sigma_n$, and the proposition follows. $\square$

ALTERNATE PROOF. The quasi-regular representation τ is weakly contained in $\oplus_{n\geq 0}\tau_n$, by "approximation of coefficients" [**Dix77**, Theorem 3.4.9]. Indeed, as δ_H, the Dirac function at H, is a cyclic vector for τ, it is enough to see that the function

$$g \mapsto \langle \tau(g)\delta_H | \delta_H \rangle = \begin{cases} 1 & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases}$$

can be pointwise approximated on G be coefficients of the τ_n's. But

$$\langle \tau_n(g)\delta_{H_n} | \delta_{H_n} \rangle = \begin{cases} 1 & \text{if } g \in H_n \\ 0 & \text{otherwise} \end{cases},$$

so

$$\lim_{n\to\infty} \langle \tau_n(g)\delta_{H_n} | \delta_{H_n} \rangle = \langle \tau(g)\delta_H | \delta_H \rangle.$$

We then have a surjection $C^*(\oplus_{n\geq 0}\tau_n) \twoheadrightarrow C^*(\tau)$, where $C^*(\rho)$ is the C^*-algebra generated by the image of ρ. The spectrum inclusions follow. $\square$

PROPOSITION E.34. *Let $\{H_n\}_{n\geq 0}$ be a descending family of finite-index subgroups of G, and set $H = \bigcup_{n\geq 0} H_n$. Let τ and τ_n be the quasi-regular representations of G on G/H and G/H_n respectively. Assume moreover that the action of G on G/H is amenable, i.e. that $\mathcal{S}(G,H,S)$ is amenable. Then*

$$\operatorname{spec} \tau = \overline{\bigcup_{n\geq 0} \operatorname{spec} \tau_n}.$$

PROOF. Let $\mathcal{S}$ be the Schreier graph $\mathcal{S}(G,H,S)$. Choose $\mu \in \operatorname{spec}\tau_n$, with a corresponding eigenvector $v : G/H_n \to \mathbb{C}$. As $\mathcal{S}$ is amenable, there is a Følner sequence $\{F_k\}$ in $\mathcal{S}$, i.e. a family of subsets of $V(\mathcal{S})$ with $|F_k|/|\partial F_k| \to 0$ as $k \to \infty$. Define now functions v_k on $\mathcal{S}$,

$$v_k(gH) = \begin{cases} v(gH_n) & \text{if } gH \in F_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\Delta_\tau v_k - \mu v_k\| \leq |\partial F_k| \cdot \|v\|$. If we let Ω be a fundamental domain of G/H_n in $\mathcal{S}$, of diameter δ, and let $\partial_\delta(F_k)$ denote the δ-neighbourhood of F_k, then we have $|\Omega| \cdot \|v_k\| \geq (|F_k| - |\partial_\delta F_k|)\|v\|$, whence

$$\left\| \Delta_\tau \frac{v_k}{\|v_k\|} - \mu \frac{v_k}{\|v_k\|} \right\| \leq \frac{|\Omega| \cdot |\partial F_k|}{|F_k| - |\partial_\delta F_k|} \to 0,$$

so $\mu \in \operatorname{spec}\tau$. $\square$

E.3.2. Spectral Measures. In parallel to the computation of spectra, interesting questions arise in relation to properties of spectral measures associated to Markovian operators. There are two approaches to spectral measures, one via a solution to the moment problem and one via trace states in von Neumann algebras.

Let $\mathfrak{G}$ be a graph, and let M be its Markovian operator. For any $x, y \in V(\mathfrak{G})$ and $n \in \mathbb{N}$ let $p^n_{x,y}$ be the probability that a simple random walk starting at x be at y after n steps. Recall that if M_v be the characteristic vector of the vertex v, then

$$p^n_{x,y} = \langle M^n \delta_x | \delta_y \rangle.$$

Define the spectral measures $\sigma_{x,y}$ by

$$p^n_{x,y} = \int_{-1}^{1} \lambda^n d\sigma_{x,y}(\lambda) \qquad \forall n \in \mathbb{N},$$

or equivalently as

$$\sigma_{x,y}(\lambda) = \langle M(\lambda)\delta_x | \delta_y \rangle,$$

where $M(\lambda)$ is the spectral decomposition of M (the operator coinciding with M on eigenfunctions whose eigenvalue is at most λ). Set $\sigma_x = \sigma_{x,x}$. These measures are called the *Kesten spectral measures*, as they were introduced in [**Kes59**].

PROPOSITION E.35. *If $\mathfrak{G}$ is connected, then all the measures $\sigma_{x,y}$ are equivalent.*

Therefore there is only one type of Kesten spectral measure, up to equivalence.

Now let N be a von Neumann algebra with a finite state τ. For any self-adjoint element $a \in N$ one can define a spectral measure τ_a by $\tau_a(\lambda) = \tau(a(\lambda))$, where $a(\lambda)$ is the spectral decomposition of a. We shall call τ_a the *von Neumann spectral measure* associated to a.

For instance, such a situation appears if N is finite-dimensional, or more generally of finite type. Another important example is when N is the von Neumann algebra generated by the left-regular representation of a group. In this case the von Neumann spectral measure is given by $\tau_a(\lambda) = \langle a(\lambda)\delta_1, \delta_1 \rangle$, where δ_1 is the Dirac function in the identity of the group. Therefore in this case the von Neumann and Kesten spectral measures of a Markovian operator coincide. A similar situation occurs if N is a hyperfinite algebra of type II_1, but there δ_1 should be replaced by any cyclic vector. If N is any algebra of type II_1 there is a canonical trace of the form $\tau(a) = \sum_i \langle ax_i, x_i \rangle$, for some (in general infinite) sequence of vectors (x_i).

Let P be a subgroup of a group G and suppose that the von Neumann algebra generated by $\rho_{G/P}$ has finite trace. Then for any symmetric system S of generators of G the von Neumann spectral measure $\tau_a = \tau_{\sum_{s\in S} s}$ can be defined. We also call τ_a the von Neumann spectral measure of the Schreier graph $\mathfrak{G} = \mathcal{S}(G, P, S)$.

In case $\mathfrak{G}$ is finite, the von Neumann spectral measure is just a "histogram", counting in any given interval the "average number of eigenvalues" that belong to it.

Suppose now that $P = \cap_{n=1}^{\infty} P_n$, where the P_n are subgroups of finite index of G. Then the sequence of finite graphs $\mathfrak{G}_n = \mathcal{S}(G, P_n, S)$ converges to the graph $\mathfrak{G}$ in the sense of [**GŻ97**], and the following statement holds:

PROPOSITION E.36 ([**GŻ97**]). *Let σ_n and σ be the Kesten spectral measures of the Schreier graphs $\mathfrak{G}_n$ and $\mathfrak{G}$ based at P_n and P respectively.*

Then we have $\sigma_n(\lambda) \Longrightarrow \sigma(\lambda)$ in the sense of weak convergence.

In case the subgroups P_n are normal and $P = 1$, we deduce from this proposition the convergence of the corresponding spectral measures τ_n to τ, since in this case $\sigma_n = \tau_n$ and $\sigma = \tau$. These results were obtained by Wolfgang Lück [**Lüc94**] and Michael Farber [**Far98**] using different methods.

It is interesting to study the conditions under which the spectral measures τ_n of finite graphs $\mathcal{S}(G, P_n, S)$ converge to some limit τ_* (which we call the *empiric* spectral measure), and, in case the von Neumann spectral measure τ is well defined for a graph $\mathcal{S}(G, P, S)$ (for instance, for a quasi-regular representation $\rho_{G/P}$ generating a von Neumann algebra of finite type), under which conditions we have $\tau = \tau_*$. Also, in the last case, when do we have $\sigma = \tau$?

Unfortunately, in our situation, the von Neumann algebra generated by $\rho_{G/P}$ is the algebra of all bounded operators so it has no a good state.

More investigations should be done in order to clarify the meaning of our computations of empiric spectral measures in the examples that follow.

E.3.3. Operator Recursions. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and suppose $\Phi : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ is an isomorphism, where the domain of Φ is a sum of $d \geq 2$ copies of $\mathcal{H}$. Let S be a finite subset of $\mathcal{U}(\mathcal{H})$, and suppose that for all $s \in S$, if we write $\Phi^{-1} s \Phi$ as an operator matrix $(s_{i,j})_{i,j\in\{1,\ldots,d\}}$ where the $s_{i,j}$ are operators in $\mathcal{H}$, then $s_{i,j} \in S \cup \{0, 1\}$.

This is precisely the setting in which we will compute the spectra of our five example groups: for G, we have $d = 2$ and $S = \{a, b, c, d\}$ with

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix},$$
$$c = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

For $\tilde{G}$, we also have $d = 2$, and $S = \{a, \tilde{b}, \tilde{c}, \tilde{d}\}$ given by

$$b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

For $\Gamma = \langle a, s\rangle$, $\overline{\Gamma} = \langle a, t\rangle$ and $\overline{\overline{\Gamma}} = \langle a, r\rangle$, we have $d = 3$ and

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix},$$
$$t = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & t \end{pmatrix}, \qquad r = \begin{pmatrix} a & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & r \end{pmatrix}.$$

Each of these operators is unitary. The families $S = \{a, b, c, d\}, \dots$ generate subgroups $G(S)$ of $\mathcal{U}(\mathcal{H})$. The choice of the isomorphism Φ defines a unitary representation of $\langle S\rangle$.

We note, however, that the expression of each operator as a matrix of operators does not uniquely determine the operator, in the sense that different isomorphisms Φ can yield non-isomorphic operators satisfying the same recursions. Even if Φ is fixed, it may happen that different operators satisfy the same recursions. We consider two types of isomorphisms in this paper: $\mathcal{H} = \ell^2(G/P)$, where Φ is derived from the ϕ defined in (36); the second case considered is $\mathcal{H} = L^2(\partial\mathcal{T})$, where $\Phi : \mathcal{H} \to \mathcal{H}^\Sigma$ is defined by $\Phi(f)(\sigma) = (f(0\sigma), \dots, f((d-1)\sigma))$, for $f \in L^2(\partial\mathcal{T})$ and $\sigma \in \partial\mathcal{T}$. There are actually uncountably many non-equivalent isomorphisms, giving uncountably many non-equivalent representations of the same group, as we indicated just before Subsection E.3.1.

E.4. Computations of Finite Spectra

Here we compute explicitly the spectra of the representations π_n described in Section E.3. For our five examples, the general principle will be the same: obtain recurrence relations on the matrices of the representation and compute eigenvalues by recurrence. We end each subsection with a computation of the spectral measure τ_Δ.

E.4.1. The group G. Recall the finite quotient G_n acts faithfully on $\{0, 1\}^n$. If we denote by a_n, b_n, c_n, d_n the permutation matrices of this representation, we have

$$a_0 = b_0 = c_0 = d_0 = (1),$$
$$a_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & c_{n-1} \end{pmatrix},$$
$$c_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & d_{n-1} \end{pmatrix}, \qquad d_n = \begin{pmatrix} 1 & 0 \\ 0 & b_{n-1} \end{pmatrix}.$$

The Hecke type operator of π_n is

$$\Delta_n = a_n + b_n + c_n + d_n = \begin{pmatrix} 2a_{n-1} + 1 & 1 \\ 1 & \Delta_{n-1} - a_{n-1} \end{pmatrix},$$

and we wish to compute its spectrum. We start by proving a slightly stronger result: define

$$Q_n(\lambda, \mu) = \Delta_n - (\lambda + 1)a_n - (\mu + 1)$$

and

$$\begin{aligned}
\Phi_0 &= 2-\mu-\lambda,\\
\Phi_1 &= 2-\mu+\lambda,\\
\Phi_2 &= \mu^2-4-\lambda^2,\\
\Phi_n &= \Phi_{n-1}^2-2(2\lambda)^{2^{n-2}} \qquad (n\geq 3).
\end{aligned}$$

LEMMA E.37. *For $n \geq 2$ we have*

$$|Q_n(\lambda,\mu)| = (4-\mu^2)^{2^{n-2}}\left|Q_{n-1}\left(\frac{2\lambda^2}{4-\mu^2},\mu+\frac{\mu\lambda^2}{4-\mu^2}\right)\right| \qquad (n\geq 2).$$

PROOF.

$$\begin{aligned}
|Q_n(\lambda,\mu)| &= \begin{vmatrix} 2a_{n-1}-\mu & -\lambda \\ -\lambda & \Delta_{n-1}-a_{n-1}-(1+\mu)\end{vmatrix}\\
&= \begin{vmatrix} 2a_{n-1}-\mu & -\lambda \\ (2a_{n-1}-\mu)\frac{-\lambda(2a_{n-1}+\mu)}{4-\mu^2} & \Delta_{n-1}-a_{n-1}-(1+\mu)\end{vmatrix}\\
&= \begin{vmatrix} 2a_{n-1}-\mu & -\lambda \\ 0 & \Delta_{n-1}-a_{n-1}-(1+\mu)-\frac{\lambda^2}{4-\mu^2}(2a_{n-1}+\mu)\end{vmatrix}\\
&= |2a_{n-1}-\mu|\cdot\left|\Delta_{n-1}-\left(1+\frac{2\lambda^2}{4-\mu^2}\right)a_{n-1}-\left(1+\mu+\frac{\mu\lambda^2}{4-\mu^2}\right)\right|;
\end{aligned}$$

now

$$|2a_{n-1}-\mu| = \begin{vmatrix} -\mu & 2 \\ 2 & -\mu \end{vmatrix}_{2^{n-1}} = (\mu^2-4)^{2^{n-2}}$$

so the lemma follows. □

LEMMA E.38. *For all n we have*

$$|Q_n| = \Phi_0\Phi_1\cdots\Phi_n.$$

PROOF. By direct computation,

$$Q_0(\lambda,\mu) = (2-\mu-\lambda), \qquad Q_1(\lambda,\mu) = \begin{pmatrix} 2-\mu & -\lambda \\ -\lambda & 2-\mu\end{pmatrix},$$

so $|Q_0| = \Phi_0$ and $|Q_1| = \Phi_0\Phi_1$.

Let us temporarily agree to write

$$\begin{aligned}
\lambda' &= \frac{2\lambda^2}{4-\mu^2} & \Phi_i &= \Phi_i(\lambda,\mu)\\
\mu' &= \mu+\frac{\mu\lambda^2}{4-\mu^2} & \Phi_i' &= \Phi_i(\lambda',\mu').
\end{aligned}$$

We will prove by recurrence that

$$\begin{aligned}
(2-\mu)\Phi_0' &= \Phi_0\Phi_1,\\
(2+\mu)\Phi_1' &= -\Phi_2,\\
(4-\mu^2)\Phi_2' &= -\Phi_3,\\
(4-\mu^2)^{2^{n-2}}\Phi_n' &= \Phi_{n+1} \qquad (n\geq 3).
\end{aligned}$$

Indeed

$$\begin{aligned}
(2-\mu)\Phi_0' &= (2-\mu)(2-\mu) - \frac{\mu\lambda^2}{2+\mu} - \frac{2\lambda^2}{2+\mu}\\
&= (2-\mu)^2 - \lambda^2 = \Phi_0\Phi_1;\\
(2+\mu)\Phi_1' &= (2+\mu)(2-\mu) - \frac{\mu\lambda^2}{2-\mu} + \frac{2\lambda^2}{2-\mu}\\
&= 4-\mu^2+\lambda^2 = -\Phi_2;\\
(4-\mu^2)\Phi_2' &= (4-\mu^2)(\mu^2-4) + 2\mu^2\lambda^2 + \frac{(\mu\lambda^2)^2}{4-\mu^2} - \frac{(2\lambda^2)^2}{4-\mu^2}\\
&= -\left[(\mu^2-4)^2 - 2\mu^2\lambda^2 + \lambda^4\right]\\
&= -\left[(\mu^2-4-\lambda^2)^2 - 8\lambda^2\right] = -\Phi_3;
\end{aligned}$$

and for $n \geq 3$,

$$\begin{aligned}
(4-\mu^2)^{2^{n-2}}\Phi_n' &= (4-\mu^2)^{2^{n-2}}(\Phi_{n-1}')^2 - (4-\mu^2)^{2^{n-2}}2(2\lambda')^{2^{n-2}}\\
&= \left((4-\mu^2)^{2^{n-3}}\Phi_{n-1}'\right)^2 - 2(4\lambda^2)^{2^{n-2}}\\
&= (\pm\Phi_n)^2 - 2(2\lambda)^{2^{n-1}} = \Phi_{n+1}.
\end{aligned}$$

Now using Lemma E.37 we have for $n \geq 3$

$$\begin{aligned}
|Q_n(\lambda,\mu)| &= (4-\mu^2)^{2^{n-2}}|Q_{n-1}(\lambda',\mu')|\\
&= (4-\mu^2)^{2^{n-2}}\Phi_0'\Phi_1'\cdots\Phi_{n-1}'\\
&= (2-\mu)\Phi_0'(2+\mu)\Phi_1'(4-\mu^2)\Phi_2'\cdots(4-\mu)^{2^{n-3}}\Phi_{n-1}'\\
&= \Phi_0\Phi_1\Phi_2\cdots\Phi_n,
\end{aligned}$$

proving the claim. ☐

PROPOSITION E.39. *For all n we have*

$$\begin{aligned}
\{(\lambda,\mu) :\ Q_n(\lambda,\mu)\ \textit{non invertible}\} &= \{(\lambda,\mu) :\ \Phi_0(\lambda,\mu)=0\} \cup \{(\lambda,\mu) :\ \Phi_1(\lambda,\mu)=0\}\\
&\cup \{(\lambda,\mu) :\ 4-\mu^2+\lambda^2+4\lambda\cos\left(\frac{2\pi j}{2^n}\right) = 0 \textit{ for some } j=1,\dots,2^{n-1}-1\}.
\end{aligned}$$

PROOF. We prove by recurrence that for all n,k with $0 \leq k \leq n-2$ we have

$$\Phi_n = \prod_{t=0}^{2^k-1}\left(\Phi_{n-k} - 2(2\lambda)^{2^{n-2-k}}\cos\left(\frac{2\pi(2t+1)}{2^{k+2}}\right)\right). \tag{39}$$

Indeed for $k=0$ equality holds trivially, and if $k>0$ we combine the terms for t and 2^k-1-t, with $t<2^{k-1}$: letting $A_{k,t}$ designate the t-th term,

$$\begin{aligned}
A_{k,t}A_{k,2^k-1-t} &= \Phi_{n-k}^2 - 4(2\lambda)^{2^{n-k-1}}\cos^2\left(\frac{2\pi(2t+1)}{2^{k+2}}\right)\\
&= \Phi_{n-k}^2 - 2\left(1+(2\lambda)^{2^{n-k-1}}\cos\left(\frac{2\pi(2t+1)}{2^{k+1}}\right)\right)\\
&= \Phi_{n-k+1} - 2(2\lambda)^{2^{n-k-1}}\cos\left(\frac{2\pi(2t+1)}{2^{k+1}}\right)\\
&= A_{k-1,t},
\end{aligned}$$

so $\prod_{t=0}^{2^k-1}A_{k,t} = \prod_{t=0}^{2^{k-1}-1}A_{k-1,t}$. Letting $k=n-2$ in (39) proves the proposition, in light of Lemma E.38. ☐

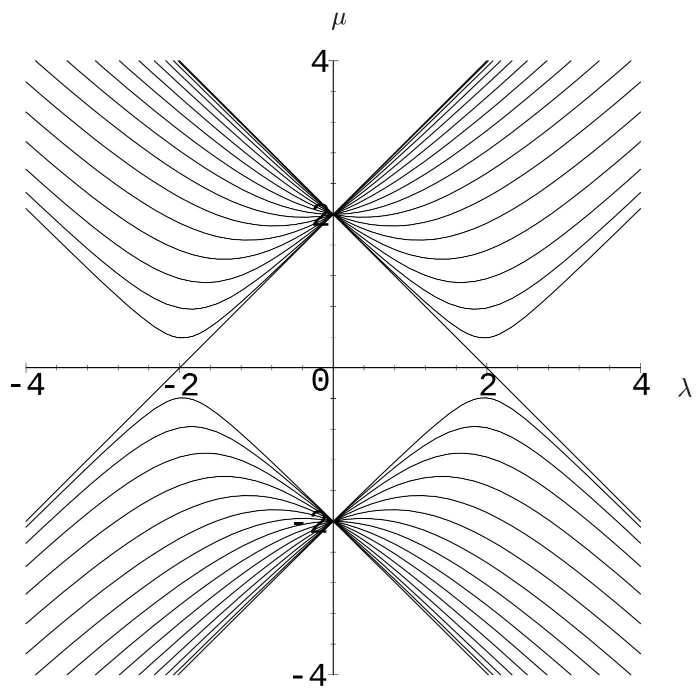


Figure E.3: The spectrum of $Q_n(\lambda, \mu)$ for G and $\tilde{G}$

In the (λ, μ) system, the spectrum of Q_n is thus a collection of 2 lines and $2^{n-1} - 1$ hyperbolæ. The spectrum of Δ_n is precisely the set of θ such that $|Q_n(-1, \theta - 1)| = 0$. From Proposition E.39 we obtain

$$\operatorname{spec}(\Delta_n) = \{1 \pm \sqrt{5 - 4\cos\phi} : \phi \in 2\pi\mathbb{Z}/2^n\} \setminus \{0, -2\}.$$

The first eigenvalues are 4; 2; $1 \pm \sqrt{5}$; $1 \pm \sqrt{5 \pm 2\sqrt{2}}$; $1 \pm \sqrt{5 \pm 2\sqrt{2 \pm \sqrt{2}}}$; etc.

The numbers of the form $\pm\sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\dots}}}$ appear as preimages of the quadratic map $z^2 - \lambda$, and after closure produce a Julia set for this map (see [**Bar88**]). In the given example this Julia set is just an interval. In the examples that follow in Subsection E.4.3 the spectra are simple transformations of Julia sets which are totally disconnected—this behaviour is explained by Lemma E.48 and the remark after its proof.

Corollary E.40. *The spectrum of π, for the group G, is*

$$\operatorname{spec}(\Delta) = [-2, 0] \cup [2, 4].$$

We now investigate the empiric spectral measure τ_Δ, as defined in Subsection E.3.2. We constructed in the previous paragraph a one-to-one map $\chi : [0, \pi] \times \{\pm 1\} \to \mathbb{R}$ defined by

$$\chi(\theta, \epsilon) = 1 + \epsilon\sqrt{5 + 4\cos\theta}.$$

The spectrum is uniformly distributed in $[0, \pi] \times \{\pm 1\}$ by Proposition E.39, and χ is by assumption a measure-preserving map, so the measure $\tau(A)$ of any subset A of $\mathbb{R}$ can be evaluated as

$$\tau(A) = \operatorname{vol}(\chi^{-1}(A)),$$

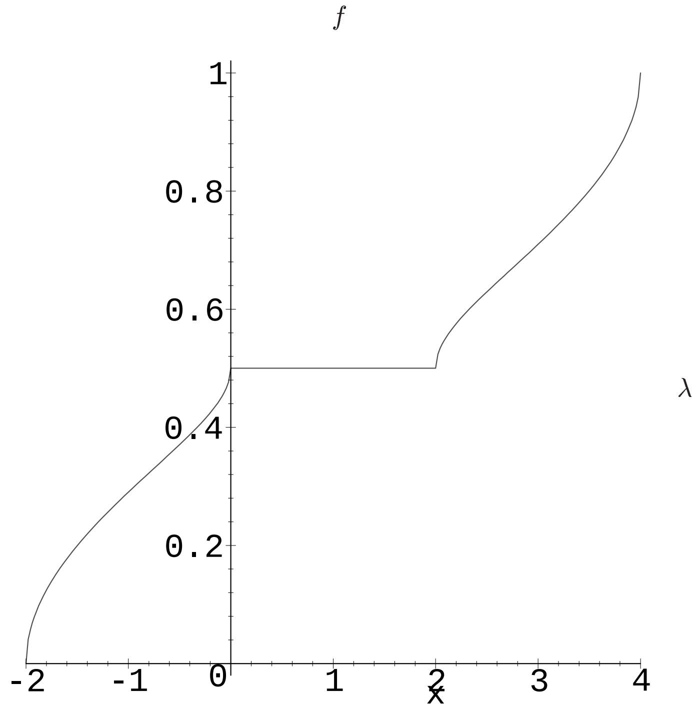


Figure E.4: The empiric spectral measure for G

where vol is the uniform measure on $[0,\pi]\times\{\pm1\}$. The measure $d\tau_\Delta(x)=g(x)dx$ we are seeking is thus given by

$$g(x)=\frac{1}{2\pi}\frac{\partial}{\partial x}\chi^{-1}(x)=\frac{1-x}{4\pi\sqrt{1-\left(\frac{(x-1)^2-5}{4}\right)^2}}.$$

The eigenvectors of Δ_n can be expressed as follows. Let λ be an eigenvalue and define by induction the sequence

$$v_1=1;\quad v_2=\lambda-3;\quad v_i=\begin{cases}\frac{(\lambda-1)v_{i-1}-v_{i-2}}{2} & \text{if } i\geq 3, i\equiv 1[2];\\ (\lambda-1)v_{i-1}-2v_{i-2} & \text{if } i\geq 3, i\equiv 0[2].\end{cases}$$

Define also by induction for all $i\geq 0$ the following ordering of Σ^i: if the ordering of Σ^{i-1} is $(\sigma_1,\dots,\sigma_{2^{i-1}})$, the ordering of Σ^i is

$$(1\sigma_1,0\sigma_1,0\sigma_2,1\sigma_2,1\sigma_3,\dots,0\sigma_{2^{i-1}},1\sigma_{2^{i-1}}).$$

Lemma E.41. *If λ is an eigenvalue of Δ_n, and $(\sigma_1,\dots,\sigma_{2^n})$ is the ordering described above, then the eigenvector corresponding to λ has value v_i on the vertex σ_i.*

Proof. Consider the Schreier graph $\mathfrak{G}_n=\mathcal{S}(G,P_n,S)$ described in Subsection E.5.1. It has vertex set $\{1,\dots,2^n\}$, a simple edge between $2i-1$ and $2i$ for all i, a double edge between $2i$ and $2i+1$ for all i, a loop at each i, and a triple loop at 1 and 2^n. The homogeneous space G_n/P_n is isomorphic to X graph through the correspondence $\sigma_i\mapsto i$. The choices of v_i clearly define an eigenvector for λ, because $v_{2^n}=0$ if λ is an eigenvalue for Δ_n. □

It is then possible to express the characteristic function of any vertex as a linear combination of the eigenvectors described above, and to explicitly compute the corresponding Kesten spectral measure. We shall not develop here this topic; but we note that the coordinates of the eigenvectors (of norm 1) vary greatly in absolute value, so the spectral measure may be very different to the empiric spectral measure.

E.4.2. The Group $\tilde{G}$. The computation of the spectrum of π_n for $\tilde{G}$ amounts to the following proposition. As before, we define $\tilde{\Delta}_n = a_n + \tilde{b}_n + \tilde{c}_n + \tilde{d}_n$, viewed as a $2^n \times 2^n$ matrix, and let

$$\tilde{Q}_n(\lambda, \mu) = \tilde{\Delta}_n - (\lambda + 1)a_n - (\mu + 2).$$

PROPOSITION E.42. *Then for all n we have*

$$\tilde{Q}_n(\lambda, \mu) = \frac{1}{2} Q_n(2\lambda, 2\mu).$$

PROOF. The proof follows by recurrence on n: first it is readily checked that $\tilde{Q}_0(\lambda, \mu) = (1 - \mu - \lambda)$; then for $n \geq 1$ we compute

$$\begin{aligned} \frac{1}{2} Q_n(2\lambda, 2\mu) &= \frac{1}{2} \begin{pmatrix} 2a_{n-1} - 2\mu & -2\lambda \\ -2\lambda & \Delta_{n-1} - a_{n-1} - (2\mu + 1) \end{pmatrix} \\ &= \begin{pmatrix} a_{n-1} - \mu & -\lambda \\ -\lambda & \frac{1}{2} Q_{n-1}(0, 2\mu) \end{pmatrix} = \begin{pmatrix} a_{n-1} - \mu & -\lambda \\ -\lambda & \tilde{Q}_{n-1}(0, \mu) \end{pmatrix} \\ &= \begin{pmatrix} a_{n-1} - \mu & -\lambda \\ -\lambda & \tilde{\Delta}_{n-1} - a_{n-1} - (\mu + 2) \end{pmatrix} = \tilde{Q}_n(\lambda, \mu). \end{aligned}$$

□

We can now obtain the spectrum of π_n by setting $\lambda = -1$ in $\tilde{Q}_n(\lambda, \mu)$ and solving for μ; in view of the previous proposition and the computations performed for G_n, we obtain:

PROPOSITION E.43. *The spectrum of $\tilde{G}_n$ is*

$$\left\{ 2 + 2\cos\left(\frac{2\pi j}{2^{n+1}}\right) \middle| j = 0, \ldots, 2^n - 1 \right\}.$$

PROOF. The spectrum consists precisely of the $\mu + 2$ such that $\tilde{Q}_n(-1, \mu) = 0$; this amounts to $Q_n(-2, 2\mu) = 0$. Now this holds when $\Phi_0(-2, 2\mu) = 0$, $\Phi_1(-2, 2\mu) = 0$ or $4 - (2\mu)^2 + 4 - 4\cos(2\pi j/2^n)(-2) = 0$ for some $j = 1, \ldots, 2^{n-1} - 1$. These give respectively $\mu = 2$, $\mu = 0$ and $\mu = \pm\sqrt{2 - 2\cos(2\pi j/2^n)}$, which after simplification yield the proposition. □

The first eigenvalues are 4; 2; $2 \pm \sqrt{2}$; $2 \pm \sqrt{2 \pm \sqrt{2}}$; $2 \pm \sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}$; etc.

COROLLARY E.44. *The spectrum of π, for the group $\tilde{G}$, is*

$$\operatorname{spec}(\Delta) = [0, 4].$$

Note that the spectrum of Δ is positive! This can never happen to regular representations [**HRV93a**], and indeed the spectrum of the regular representation of $\tilde{G}$ is $[-4, 4]$, since it contains $[0, 4]$ and is symmetrical about 0 (as the Cayley graph $\mathcal{C}(\tilde{G}, S)$ is bipartite).

Again we may compute the empiric spectral measure τ_Δ. Recall that there is a measure-preserving map $\chi : [0, \pi] \to \mathbb{R}$ given by $\theta \mapsto 2 + 2\cos\theta$, where $[0, \pi]$ has the Lebesgue measure. The measure $d\tau_\Delta = g(x)dx$ we are seeking is then associated to

$$g(x) = \frac{1}{\pi} \frac{\partial}{\partial x} \chi^{-1}(x) = \frac{1}{\pi\sqrt{4x - x^2}}.$$

E.4.3. The Group Γ. Recall that the finite quotient Γ_n acts faithfully on $\{0,1,2\}^n$. Denote by a_n and s_n the matrices of the action. We have

$$a_0 = s_0 = (1),$$
$$a_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad s_n = \begin{pmatrix} a_{n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_{n-1} \end{pmatrix}.$$

The combinatorial Laplacian of Γ_n is

$$\Delta_n = a_n + a_n^{-1} + s_n + s_n^{-1} = \begin{pmatrix} a_{n-1} + a_{n-1}^{-1} & 1 & 1 \\ 1 & a_{n-1} + a_{n-1}^{-1} & 1 \\ 1 & 1 & s_{n-1} + s_{n-1}^{-1} \end{pmatrix}.$$

For ease of notation, let us define operators

$$A_n = a_n + a_n^{-1}, \qquad S_n = s_n + s_n^{-1},$$
$$Q_n(\lambda, \mu) = S_n + \lambda A_n - \mu,$$

and polynomials

$$\alpha = 2 - \mu + \lambda, \qquad \beta = 2 - \mu - \lambda,$$
$$\gamma = \mu^2 - \lambda^2 - \mu - 2, \qquad \delta = \mu^2 - \lambda^2 - 2\mu - \lambda.$$

Lemma E.45. *We have*

$$|Q_0(\lambda,\mu)| = 2 + 2\lambda - \mu = \alpha + \lambda,$$
$$|Q_1(\lambda,\mu)| = (2 + 2\lambda - \mu)(2 - \lambda - \mu)^2 = (\alpha + \lambda)\beta^2,$$
$$|Q_n(\lambda,\mu)| = (\alpha\beta\gamma^2)^{3^{n-2}} \left| Q_{n-1}\left(\frac{\lambda^2\beta}{\alpha\gamma}, \mu + \frac{2\lambda^2\delta}{\alpha\gamma}\right) \right| \quad (n \geq 2).$$

Proof. We compute the determinant:

$$|Q_n(\lambda,\mu)| = |S_n + \lambda A_n - \mu| = \begin{vmatrix} A_{n-1} - \mu & \lambda & \lambda \\ \lambda & 2 - \mu & \lambda \\ \lambda & \lambda & S_{n-1} - \mu \end{vmatrix}$$
$$= \begin{vmatrix} 2 - \mu - \lambda & \lambda + \mu - A_{n-1} & 0 \\ \lambda & A_{n-1} - \mu & \lambda \\ \lambda & \lambda & S_{n-1} - \mu \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 0 & 0 \\ * & (2-\mu)A_{n-1} + (\mu^2 - \lambda^2 - 2\mu) & \lambda \\ * & \lambda A_{n-1} + \lambda(2 - 2\lambda - 2\mu) & S_{n-1} - \mu \end{vmatrix}$$
$$= \left| S_{n-1} - \mu - \frac{\lambda^2(1 - 2\lambda - \mu + A_{n-1})}{(1-\mu)A_{n-1} + \mu^2 - \lambda^2} \right| \cdot |(1-\mu)A_{n-1} + \mu^2 - \lambda^2|$$
$$= \left| S_{n-1} - \mu + \lambda^2 \frac{\beta A_{n-1} + 2\delta}{\alpha\gamma} \right| (\alpha\beta\gamma^2)^{3^{n-2}},$$

which completes the proof of the lemma, using the easily verified equations

$$\frac{1}{\pi A_n + \rho} = \frac{\pi A_n - \pi - \rho}{(2\pi + \rho)(\pi - \rho)},$$
$$|\pi A_n + \rho| = \begin{vmatrix} \rho & \pi & \pi \\ \pi & \rho & \pi \\ \pi & \pi & \rho \end{vmatrix}_{3^n} = \left((\pi - \rho)^2(2\pi + \rho)\right)^{3^{n-1}}$$

valid for all scalar π and ρ. $\square$

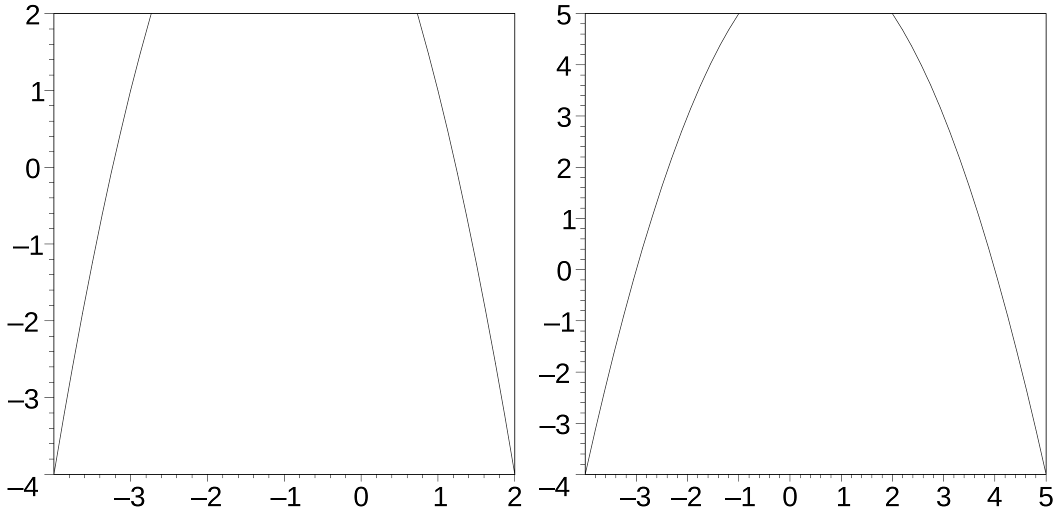


Figure E.5: The Functions F for Γ and $\overline{\Gamma}$

Consider now the quadratic forms

$$H_\theta = \mu^2 - \lambda\mu - 2\lambda^2 - 2 - \mu + \theta\lambda,$$

and the function $F : [-4, 5] \to [-4, 5]$ (see Figure E.5 left) given by

$$F(\theta) = 4 - 2\theta - \theta^2.$$

Let $X_2 = \{-1\}$, and iteratively define $X_n = F^{-1}(X_{n-1})$ for all $n \geq 3$. Note that $|X_n| = 2^{n-2}$.

Lemma E.46. *We have for all $n \geq 2$ the factorization*

$$|Q_n(\lambda, \mu)| = (2 + 2\lambda - \mu)(2 - \lambda - \mu)^{3^{n-1}+1} \prod_{\substack{2 \leq m \leq n \\ \theta \in X_m}} H_\theta^{3^{n-m}+1}. \tag{40}$$

Proof. Let us write temporarily

$$\lambda' = \frac{\lambda^2\beta}{\alpha\gamma}, \qquad \mu' = \mu + \frac{2\lambda^2\delta}{\alpha\gamma},$$

and $P' = P(\lambda', \mu')$ for P in $\{\alpha, \beta, \gamma, \delta, H_\theta\}$. Then we have

$$\alpha' + \lambda' = \frac{\beta(\alpha + \lambda)}{\alpha}, \qquad \beta' = \frac{\beta H_{-1}}{\gamma},$$
$$H_\theta(\lambda', \mu') = \frac{\beta \prod_{\nu \in F^{-1}(\theta)} H_\nu(\lambda, \mu)}{\alpha\gamma}.$$

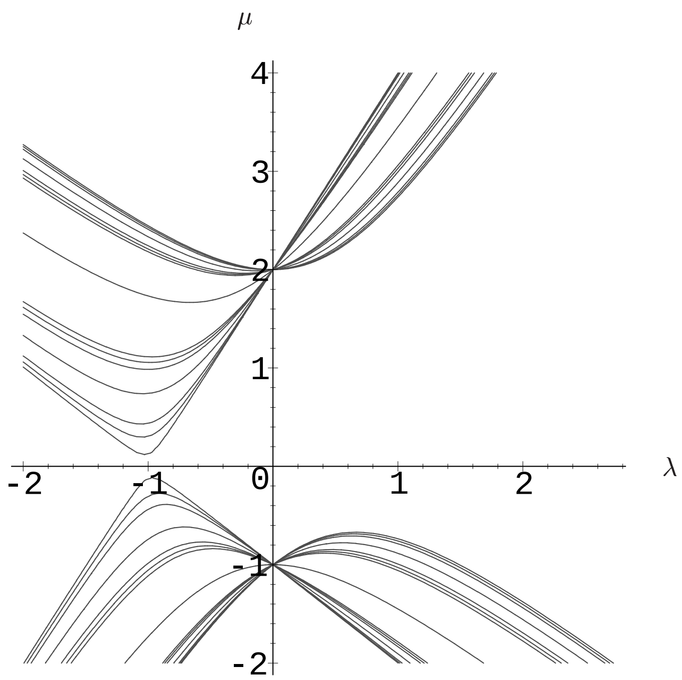


Figure E.6: The spectrum of $Q_n(\lambda, \mu)$ for Γ

Now, first $Q_2(\lambda, \mu) = (\alpha + \lambda)\beta^4 H_{-1}^2$ as claimed; then by induction, using Lemma E.45, we have for $n \geq 3$

$$\begin{aligned}
|Q_n(\lambda, \mu)| &= (\alpha\beta\gamma 2)^{3^{n-2}} |Q_{n-1}(\lambda', \mu')| \\
&= (\alpha\beta\gamma^2)^{3^{n-2}} (\alpha' + \lambda')\beta'^{3^{n-2}+1} \prod_{\substack{2 \leq m \leq n-1 \\ \theta \in X_m}} H_\theta'^{3^{n-1-m}+1} \\
&= (\alpha\beta\gamma^2)^{3^{n-2}} \frac{\beta(\alpha + \lambda)}{\alpha} \left(\frac{\beta H_{-1}}{\gamma}\right)^{3^{n-2}+1} \left(\frac{\beta}{\alpha\gamma}\right)^{3^{n-2}-1} \prod_{\substack{2 \leq m \leq n \\ \theta \in X_m}} H_\theta^{3^{n-m}+1} \\
&= (\alpha + \lambda)\beta^{3^{n-1}+1} \prod_{\substack{2 \leq m \leq n \\ \theta \in X_m}} H_\theta^{3^{n-m}+1}
\end{aligned}$$

as claimed. □

Thus, according to the previous proposition, the spectrum of Q_n is a collection of two lines and $2^{n-1} - 1$ hyperbolæ H_θ with $\theta \in X_2 \sqcup X_3 \sqcup \cdots \sqcup X_n$. The spectrum of Δ_n is obtained by solving $|Q_n(1, \mu)| = 0$, as given in the following proposition.

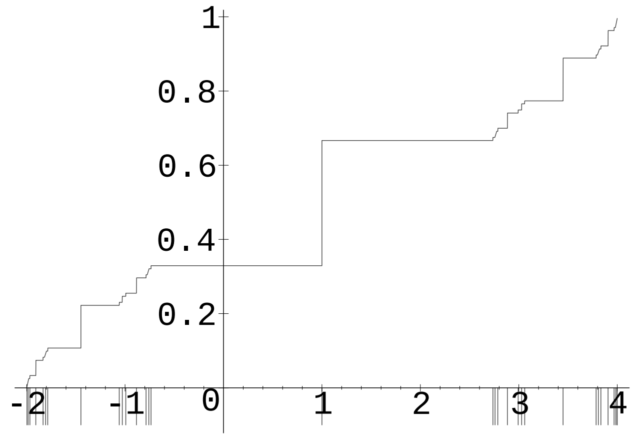


Figure E.7: The Cantor-Set spectrum of Δ_n for Γ

Proposition E.47. *Let $\pi_\pm : [-4,5] \to [-2,4]$ be defined by $\pi_\pm(\theta) = 1 \pm \sqrt{5-\theta}$. Then*

$$\operatorname{spec}\Delta_0 = \{4\};$$
$$\operatorname{spec}\Delta_1 = \{1,4\};$$
$$\operatorname{spec}\Delta_n = \{1,4\} \cup \bigcup_{2\le m\le n} \pi_\pm(X_m) \qquad (n \ge 2).$$

Proof. Solving $H_\theta(1,\mu) = 0$ gives $\mu = \pi_\pm(\theta)$. The result then follows from the factorization of $|Q_n|$ as a product of H_θ's given by Lemma E.51. □

Lemma E.48. *Let $F : [a,b] \to \mathbb{R}$ be a smooth unimodal map with negative Schwartzian derivative (for instance, a quadratic map), and $F(a) = F(b) = a$. Choose $\xi \in [a,b]$, and form*

$$K = \bigcap_{n=0}^{\infty} F^{-n}([a,b]), \qquad L = \overline{\bigcup_{n=0}^{\infty} F^{-n}(\{\xi\})}.$$

Then the following cases may occur:

- $\mathbf{F([a,b]) \subset [a,b]}$: *Then $K = [a,b]$.*
- $\mathbf{F([a,b]) = [a,b]}$: *Then $K = L = [a,b]$.*
- $\mathbf{F([a,b]) \supset [a,b]}$: *Then K is a Cantor set of null Lebesgue measure. If $\xi \in K$, then $L = K$, otherwise $L = K \sqcup C$, where C is a countable set of isolated points accumulating on K.*

Proof. In the first case: K is clearly $[a,b]$, because $F^{-1}([a,b]) = [a,b]$.

Now assume that $F([a,b]) \supseteq [a,b]$, and define two continuous maps $g_1, g_2 : [a,b] \to [a,b]$ such that $F^{-1}(x) = \{g_1(x), g_2(x)\}$. Then $\Pi = \langle g_1, g_2 \rangle$ is a free semigroup, because g_1 and g_2 have disjoint image-interiors.

In the second case: K is obviously all $[a,b]$, because $F^{-1}([a,b]) = [a,b]$. Clearly Π-orbit of ξ, is dense in $[a,b] = g_1([a,b]) \cup g_2([a,b])$, so its closure L is $[a,b]$.

In the last case: we first show that $K \subseteq L$. Since $K = g_1(K) \sqcup g_2(K)$, every point in K is specified by an infinite sequence of g_i's applied to some point. Since the g_i's are contracting, the choice of that point is unimportant — hence we may choose ξ, and this proves the claim. If $\xi \in K$, then clearly $L \subseteq K$ and we are done. Otherwise, set $C = \cup F^{-n}(\{\xi\})$, which is a countable set of isolated points. Clearly K and C are disjoint (else ξ would be in K). Finally C accumulates on the infinite orbits under $\langle g_1, g_2\rangle$, hence on K.

Let $O = [a,b] \setminus F^{-1}([a,b])$. Then O disconnects $[a,b]$, and $F^{-1}(O)$ disconnects each of the connected components of $[a,b] \setminus O$, etc., so K is disconnected. Take any point $x \in K$; then for all $n \in \mathbb{N}$ we may write $x = w(x_n)$ for some $x_n \in K$ and some word $w \in \Pi$ of length n. Taking some $x'_n \in K$ close to x_n and letting n tend to infinity gives a sequence in K converging to x. Since K is clearly closed, it is perfect, so is a Cantor set. Finally $m(K) = \lim_{n\to\infty}(1 - m(O))^n = 0$, where m denotes the Lebesgue measure. □

Let us make a few remarks concerning this lemma:

- The behaviour of L in the first case seems unpredictable. It is fortunately not needed for our purposes.
- The second case arises in connection with G and $\tilde{G}$, the third (with $\xi \notin K$) with Γ and (with $\xi \in K$) with $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$.
- Much more is known on the forward and inverse orbits of points under a unimodal map. See for instance [**dMvS93**, pages 10 and 327–351] for more information.
- K is the Julia set of the dynamical system F, i.e. the closure of the set of points whose (forward) orbit does not diverge to infinity. Indeed K consists precisely in those points whose orbit remains in $[a,b]$. Examples of Julia sets of a similar nature (occurring in the study of the quadratic map), with nested square-root expressions, appear in [**Bar88**, page 277] — see also the bibliography there.

The previous lemma gives the structure of the spectrum of Γ:

Corollary E.49. *The spectrum of π for the group Γ is the closure of the set*

$$\left\{\begin{array}{rl} 4 & (\mu = 0)\\ 1 & (\mu = \frac{1}{3})\\ 1 \pm \sqrt{6} & (\mu = \frac{1}{9})\\ 1 \pm \sqrt{6 \pm \sqrt{6}} & (\mu = \frac{1}{3^3})\\ 1 \pm \sqrt{6 \pm \sqrt{6 \pm \sqrt{6}}} & (\mu = \frac{1}{3^4})\\ \dots & \end{array}\right\}.$$

It is the union of a Cantor set of null Lebesgue measure that is symmetrical about 1, *and a countable collection of isolated points supporting the empiric spectral measure, which has the values indicated as μ.*

E.4.4. The Group $\overline{\Gamma}$. Recall the finite quotient Γ_n acts faithfully on $\{0,1,2\}^n$. Denote by a_n and t_n the matrices of this action. We have

$$a_0 = t_0 = (1),$$

$$a_n = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \qquad t_n = \begin{pmatrix} a_{n-1} & 0 & 0\\ 0 & a_{n-1} & 0\\ 0 & 0 & t_{n-1}\end{pmatrix}.$$

The combinatorial Laplacian of Γ_n is

$$\Delta_n = a_n + a_n^{-1} + t_n + t_n^{-1} = \begin{pmatrix} a_{n-1} + a_{n-1}^{-1} & 1 & 1\\ 1 & a_{n-1} + a_{n-1}^{-1} & 1\\ 1 & 1 & t_{n-1} + t_{n-1}^{-1}\end{pmatrix}.$$

For ease of notation, let us define operators

$$A_n = a_n + a_n^{-1}, \qquad T_n = t_n + t_n^{-1},$$
$$Q_n(\lambda,\mu) = T_n + \lambda A_n - \mu,$$

and polynomials

$$\alpha = 2-\mu+\lambda, \qquad \beta = 2-\mu-\lambda,$$
$$\gamma = 1+\mu+\lambda, \qquad \delta = 1+\mu-\lambda.$$

LEMMA E.50. *We have*

$$|Q_0(\lambda,\mu)| = \alpha+\lambda,$$
$$|Q_1(\lambda,\mu)| = (\alpha+\lambda)\beta^2,$$
$$|Q_n(\lambda,\mu)| = (\gamma\delta)^{2\cdot 3^{n-2}}(\alpha\beta)^{3^{n-2}}\left|Q_{n-1}\left(\frac{-2\lambda^2}{\alpha\delta}, \mu + \frac{2\lambda^2(\mu-\lambda-1)}{\alpha\delta}\right)\right| \quad (n\geq 2).$$

PROOF. We compute the determinant:

$$|Q_n(\lambda,\mu)| = |T_n+\lambda A_n-\mu| = \begin{vmatrix} A_{n-1}-\mu & \lambda & \lambda \\ \lambda & A_{n-1}-\mu & \lambda \\ \lambda & \lambda & T_{n-1}-\mu \end{vmatrix}$$
$$= \begin{vmatrix} A_{n-1}-\mu-\lambda & 0 & 0 \\ \lambda & A_{n-1}-\mu+\lambda & \lambda \\ \lambda & 2\lambda & T_{n-1}-\mu \end{vmatrix}$$
$$= \left|T_{n-1}-\mu-\frac{2\lambda^2}{A_{n-1}-\mu+\lambda}\right|\cdot|A_{n-1}-\mu+\lambda|\cdot|A_{n-1}-\mu-\lambda|$$
$$= \left|T_{n-1}-\mu-2\lambda^2\frac{A_{n-1}+\mu-\lambda-1}{(2-\mu+\lambda)(1+\mu-\lambda)}\right|\cdot|A_{n-1}-\mu+\lambda|\cdot|A_{n-1}-\mu-\lambda|$$

which completes the proof of the lemma, using the easily verified equations

$$\frac{1}{A_n-\theta} = \frac{A_n+\theta-1}{(2-\theta)(1+\theta)},$$
$$|A_n-\theta| = \begin{vmatrix} -\theta & 1 & 1 \\ 1 & -\theta & 1 \\ 1 & 1 & -\theta \end{vmatrix}_{3^{n-1}} = \big((\theta+1)^2(2-\theta)\big)^{3^{n-1}}$$

valid for all scalar θ. ☐

Consider again the quadratic forms

$$H_\theta = \mu^2-\lambda\mu-2\lambda^2-2-\mu+\theta\lambda,$$

and consider the real function (see Figure E.5 right)

$$F(\theta) = \frac{12+\theta-\theta^2}{2}.$$

Let $X_3=\{2\}$, $Y_3=\{-1\}$ and iteratively define $X_n = F^{-1}(X_{n-1})$ and $Y_n = F^{-1}(Y_{n-1})$ for all $n\geq 4$. Note that $|X_n| = |Y_n| = 2^{n-3}$.

LEMMA E.51. *We have for all $n\geq 2$ the factorization*

(41)

$$|Q_n(\lambda,\mu)| = (\alpha+\lambda)\beta^{3^{n-2}+1}\gamma^{3^{n-1}-1}(\delta-\lambda)^{3^{n-2}-1}\prod_{\substack{3\leq m\leq n\\ \theta\in X_m}} H_\theta^{3^{n-m}+1}\prod_{\substack{3\leq m<n\\ \theta\in Y_m}} H_\theta^{3^{n-m}-1}\prod_{\theta\in X_{n+1}} H_\theta^2.$$

PROOF. Let us write temporarily

$$\lambda' = \frac{-2\lambda^2}{\alpha\delta}, \qquad \mu' = \mu + \frac{2\lambda^2(\mu-\lambda-1)}{\alpha\delta},$$

and $P' = P(\lambda',\mu')$ for P in $\{\alpha,\beta,\gamma,\delta,H_\theta\}$. Then we have

$$\alpha'+\lambda' = \frac{\beta(\alpha+\lambda)}{\alpha}, \qquad \beta' = \frac{-H_2}{\delta}, \qquad \gamma' = \frac{\gamma(\delta-\lambda)}{\delta}, \qquad \delta'-\lambda' = \frac{-H_{-1}}{\alpha},$$
$$H_\theta(\lambda',\mu') = \frac{\prod_{\nu\in F^{-1}(\theta)} H_\nu(\lambda,\mu)}{\alpha\delta}.$$

Now, first $Q_2(\lambda,\mu) = (\alpha+\lambda)\beta^2\gamma^2H_2^2$ as claimed; then by induction, using Lemma E.50, we have for $n\geq 3$

$$\begin{aligned}
|Q_n(\lambda,\mu)| &= (\gamma\delta)^{2\cdot 3^{n-2}}(\alpha\beta)^{3^{n-2}}\,|Q_{n-1}(\lambda',\mu')| \\
&= (\gamma\delta)^{2\cdot 3^{n-2}}(\alpha\beta)^{3^{n-2}}(\alpha'+\lambda'){\beta'}^{3^{n-3}+1}{\gamma'}^{3^{n-2}-1}(\delta'-\lambda')^{3^{n-3}-1} \\
&\qquad \prod_{\substack{3\leq m\leq n-1\\ \theta\in X_m}} {H'_\theta}^{3^{n-1-m}+1} \prod_{\substack{3\leq m<n-1\\ \theta\in Y_m}} {H'_\theta}^{3^{n-1-m}-1} \prod_{\theta\in X_n} {H'_\theta}^2 \\
&= (\gamma\delta)^{2\cdot 3^{n-2}}(\alpha\beta)^{3^{n-2}}\frac{\beta(\alpha+\lambda)}{\alpha}\left(\frac{-H_2}{\delta}\right)^{3^{n-3}+1}\left(\frac{\gamma(\delta-\lambda)}{\delta}\right)^{3^{n-2}-1}\left(\frac{-H_{-1}}{\alpha}\right)^{3^{n-3}-1} \\
&\qquad (\alpha\delta)^{-2\cdot 3^{n-3}}\prod_{\substack{4\leq m\leq n\\ \theta\in X_m}} H_\theta^{3^{n-m}+1} \prod_{\substack{4\leq m<n\\ \theta\in Y_m}} H_\theta^{3^{n-m}-1} \prod_{\theta\in X_{n+1}} H_\theta^2. \\
&= (\alpha+\lambda)\beta^{3^{n-2}+1}\gamma^{3^{n-1}-1}(\delta-\lambda)^{3^{n-2}-1}\prod_{\substack{3\leq m\leq n\\ \theta\in X_m}} H_\theta^{3^{n-m}+1} \prod_{\substack{3\leq m<n\\ \theta\in Y_m}} H_\theta^{3^{n-m}-1} \prod_{\theta\in X_{n+1}} H_\theta^2
\end{aligned}$$

as claimed. □

Thus, according to the previous proposition, the spectrum of Q_n is a collection of 4 lines and $(2^{n-1}-1)+(2^{n-3}-1)$ hyperbolæ H_θ with $\theta\in X_3\sqcup\cdots\sqcup X_{n+1}\sqcup Y_3\sqcup\cdots\sqcup Y_{n-1}$. The spectrum of Δ_n is obtained by solving $|Q_n(1,\mu)|=0$, as given in the following proposition.

PROPOSITION E.52. *Let $\pi_\pm : [-4,5]\to[-2,4]$ be defined by $\pi_\pm(\theta) = 1\pm\sqrt{5-\theta}$. Then*

$$\begin{aligned}
\operatorname{spec}\Delta_0 &= \{4\}; \\
\operatorname{spec}\Delta_1 &= \{1,4\}; \\
\operatorname{spec}\Delta_n &= \{-2,1,4\}\cup\bigcup_{3\leq m\leq n+1}\pi_\pm(X_m)\cup\bigcup_{3\leq m\leq n-1}\pi_\pm(Y_m) \qquad (n\geq 2).
\end{aligned}$$

PROOF. Solving $H_\theta(1,\mu)=0$ gives $\mu=\pi_\pm(\theta)$. The result then follows from the factorization of $|Q_n|$ as a product of H_θ's given by Lemma E.51. □

COROLLARY E.53. *The spectrum of π for the group $\overline{\Gamma}$ is the closure of the set*

$$\left\{\begin{array}{rl}
4 & (\mu=0) \\
-2 & (\mu=\frac{1}{3}) \\
1 & (\mu=\frac{2}{9}) \\
1\pm\sqrt{\frac{9\pm 3}{2}} & (\mu=\frac{2}{27}) \\
1\pm\sqrt{\frac{9\pm\sqrt{45\pm 4\cdot 3}}{2}} & (\mu=\frac{2}{3^4}) \\
1\pm\sqrt{\frac{9\pm\sqrt{45\pm 4\sqrt{45\pm 4\cdot 3}}}{2}} & (\mu=\frac{2}{3^5}) \\
\dots &
\end{array}\right\}.$$

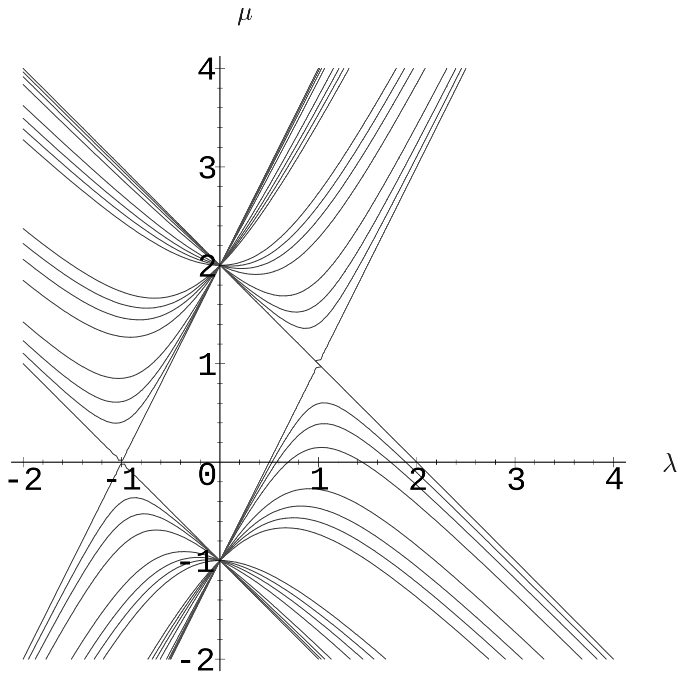


Figure E.8: The spectrum of $Q_n(\lambda, \mu)$ for $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$

It is a Cantor set of null Lebesgue measure that is symmetrical about 1*. The empiric spectral measure is concentrated on the above algebraic numbers and has the values indicated as* μ*.*

E.4.5. The Group $\overline{\overline{\Gamma}}$. Although $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ greatly differ in structure—the former is virtually torsion-free while the latter is torsion—their representations π_n have the same spectrum:

PROPOSITION E.54. *The spectrum of $\overline{\overline{\Gamma}}$ is the same as that of $\overline{\Gamma}$; that is, a Cantor set symmetrical about* 1 *and spanning from* -2 *to* 4*.*

PROOF. It suffices to note that for all n we have $r_n + r_n^{-1} = t_n + t_n^{-1}$; this follows by induction on n. □

E.5. Schreier Graphs

Recall from Section E.3 the definition of a Schreier graph. In the general setting of a group G acting level-transitively on a tree Σ^*, and a subgroup P defined as the stabilizer of an infinite path, the quotient space G/P is naturally identified with an orbit, i.e. a countable subset, of $\Sigma^{\mathbb{N}}$. For the finite quotients G_n, the space G_n/P_n is identified with Σ^n. We set $\mathfrak{G}_n$ denote the Schreier graph associated to this homogeneous space. Due to the fractal (or recursive) nature of the five examples we study, there are simple local rules producing $\mathfrak{G}_{n+1}$ from $\mathfrak{G}_n$, the limit of this process being the Schreier graph of G/P. We describe these rules for our examples: $G, \tilde{G}, \Gamma, \overline{\Gamma}, \overline{\overline{\Gamma}}$.

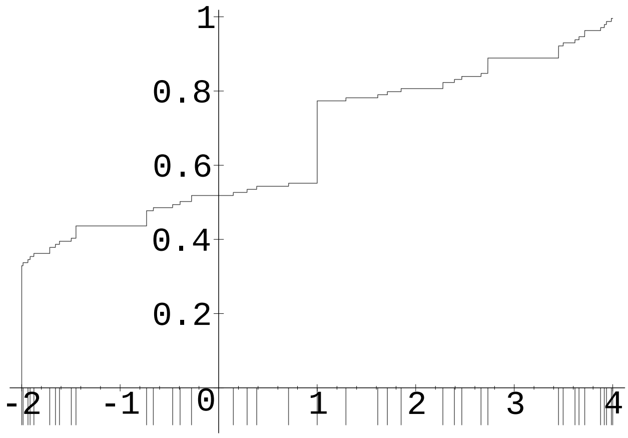


Figure E.9: The Cantor-Set spectrum of Δ_n for $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$

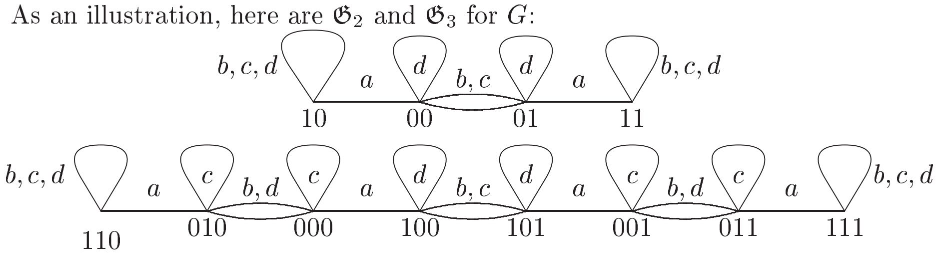
E.5.1. $\mathcal{S}(G,P,S)$**.** We describe the graphs $\mathfrak{G}_n = \mathcal{S}(G_n, P_n, S)$. They will be with edges labeled by S (and not oriented, because all $s \in S$ are involutions) and vertices labeled by Σ^n, where $\Sigma = \{0,1\}$.

First, it is clear that $\mathfrak{G}_0$ is a graph on one vertex, labeled by the empty sequence $\emptyset$, and four loops at this vertex, labeled by a,b,c,d. Next, $\mathfrak{G}_1$ has two vertices, labeled by 0 and 1; an edge labeled a between them; and three loops at 0 and 1 labeled by b,c,d.

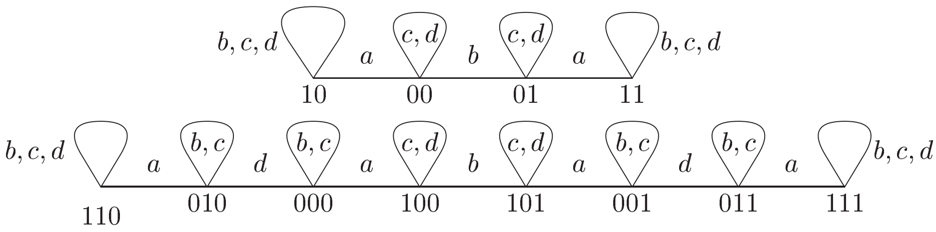
Now given $\mathfrak{G}_n$, for some $n \geq 1$, perform on it the following transformation: replace the edge-labels b by d, d by c, c by b; replace the vertex-labels σ by 1σ; and replace all edges labeled by a connecting σ and τ by: edges from 1σ to 0σ and from 1τ to 0τ, labeled a; two edges from 0σ to 0τ labeled b and c; and loops at 0σ and 0τ labeled d. We claim the resulting graph is $\mathfrak{G}_{n+1}$.

To prove the claim, it suffices to check that the letters on the edge-labels act as described on the vertex-labels. If $b(\sigma) = \tau$, then $d(1\sigma) = 1\tau$, and similarly for c and d; this explains the cyclic permutation of the labels b,c,d. The other substitutions are verified similarly.

As an illustration, here are $\mathfrak{G}_2$ and $\mathfrak{G}_3$ for G:



The graphs $\mathfrak{G}_2$ and $\mathfrak{G}_3$ for $\tilde{G}$ are similar:



E.5.2. $\mathcal{S}(\Gamma, P, S)$**.** Recall that for Γ we take $\Sigma = \{0, 1, 2\}$. Let us write $\mathfrak{G}_n = \mathcal{S}(\Gamma, P_n, S)$. First, $\mathfrak{G}_0$ has one vertex, labeled by the empty sequence $\emptyset$, and four loops, labeled a, a^{-1}, t, t^{-1}. Next, $\mathfrak{G}_1$ has three vertices, labeled $0, 1, 2$, cyclically connected by a triangle labeled a, a^{-1}, and with two loops at each vertex, labeled t, t^{-1}. In the pictures only geometrical edges, in pairs $\{a, a^{-1}\}$ and $\{t, t^{-1}\}$, are represented.

Now given $\mathfrak{G}_n$, for some $n \geq 1$, perform on it the following transformation: replace the vertex-labels σ by 2σ; replace all triangles labeled by a, a^{-1} connecting ρ, σ, τ by: three triangles labeled by a, a^{-1} connecting respectively $0\rho, 1\rho, 2\rho$ and $0\sigma, 1\sigma, 2\sigma$ and $0\tau, 1\tau, 2\tau$; a triangle labeled by t, t^{-1} connecting $0\rho, 0\sigma, 0\tau$; and loops labeled by t, t^{-1} at $1\rho, 1\sigma, 1\tau$. We claim the resulting graph is $\mathfrak{G}_{n+1}$.

As above, it suffices to check that the letters on the edge-labels act as described on the vertex-labels. If $a(\rho) = \sigma$ and $t(\rho) = \tau$, then $t(0\rho) = 0\sigma$, $t(1\rho) = 1\rho$ and $t(2\sigma) = 2\tau$. The verification for a edges is even simpler.

E.5.3. Substitutional graphs. The three Schreier graphs presented in the previous subsection are special cases of *substitutional graphs*, which we define below.

Substitutional graphs were introduced in the late 70's to describe growth of multicellular organisms. They bear a strong similarity to L-systems [**RS80**], as was noted by Mikhael Gromov [**Gro84**]. Another notion of graph substitution has been studied by [**Pre98**], where he had the same convergence preoccupations as us.

Let us make a conventions in this section: all graphs $\mathfrak{G} = (V(\mathfrak{G}), E(\mathfrak{G}))$ shall be *labeled*, i.e. endowed with a map $E(\mathfrak{G}) \to C$ for a fixed set C of colours, and *pointed*, i.e. shall have a distinguished vertex $* \in V(\mathfrak{G})$. A *graph embedding* $\mathfrak{G}' \hookrightarrow \mathfrak{G}$ is just an injective map $E(\mathfrak{G}') \to E(\mathfrak{G})$ preserving the adjacency operations.

Definition E.55. *A* substitutional rule *is a tuple* $(U, R_1, \dots, R_n)$*, where* U *is a finite* d*-regular edge-labeled graph, called the* axiom*, and each* R_i *is a rule of the form* $X_i \to Y_i$*, where* X_i *and* Y_i *are finite edge-labeled graphs. The graphs* X_i *are required to have no common edge. Furthermore, there is a inclusion, written* ι_i*, of the vertices of* X_i *in the vertices of* Y_i*; the degree of* $\iota_i(x)$ *is the same as the degree of* x *for all* $x \in V(X_i)$*, and all vertices of* Y_i *not in the image of* ι_i *have degree* d*.*

Given a substitutional rule, one sets $\mathfrak{G}_0 = U$ and constructs iteratively $\mathfrak{G}_{n+1}$ from $\mathfrak{G}_n$ by listing all embeddings of all X_i in $\mathfrak{G}_n$ (they are disjoint), and replacing them by the corresponding Y_i. If the base point $*$ of $\mathfrak{G}_n$ is in a graph X_i, the base point of $\mathfrak{G}_{n+1}$ will be $\iota_i(*)$.

Note that this expansion operation preserves the degree, so $\mathfrak{G}_n$ is a d-regular finite graph for all n. We are interested in fixed points of this iterative process.

For any $R \in \mathbb{N}$, consider the balls $B_n(*, R)$ of radius R at the base point $*$ in $\mathfrak{G}_n$. Since there is only a finite number of rules, there is an infinite sequence $n_0 < n_1 < \dots$ such that the balls $B_{n_i}(*, R) \subseteq \mathfrak{G}_{n_i}$ are all isomorphic. We consider $\mathfrak{G}$, a limit graph in the sense of [**GŻ97**] (the limit exists), and call it a *substitutional graph*.

Note that in case some rule $X_i \to Y_i$ does not satisfy $X_i \subset Y_i$, there may be more than one limit graph — this is the case in the following example (where there are in fact three limit graphs, according to whether the next-to-rightmost edge is b, c or b, d or c, d).

Theorem E.56. *The following four substitutional rules describe the Schreier graph of* G*:*

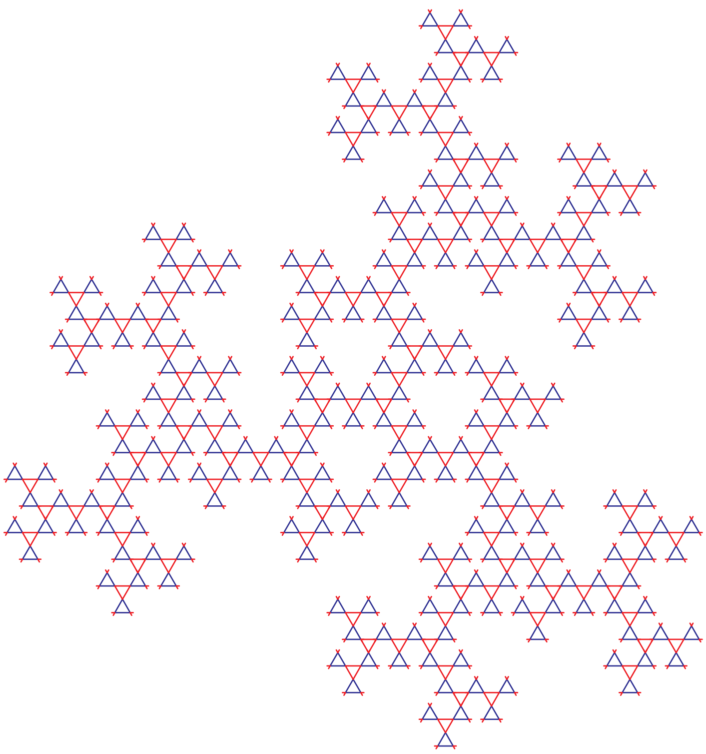
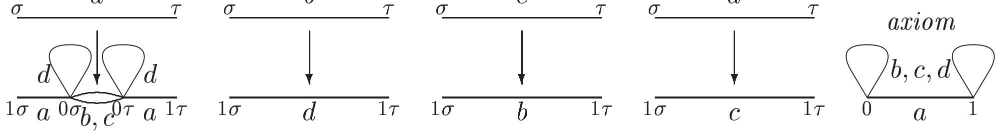


Figure E.10: The Schreier Graph of Γ_6. The red and blue edges represent the generators s and a.



where the vertex inclusions are given by $\sigma \mapsto 1\sigma$ *and* $\tau \mapsto 1\tau$*. The base point is the vertex* $111\dots$.

Proof. Consider the Schreier graph $\mathfrak{G}_n$ associated to the action of G on Σ^n, the n-th level of the tree $\mathcal{T}_\Sigma$. The vertex set of $\mathfrak{G}_n$ is Σ^n, and its edges are described by the action of G. Note first that the axiom is $\mathfrak{G}_0$.

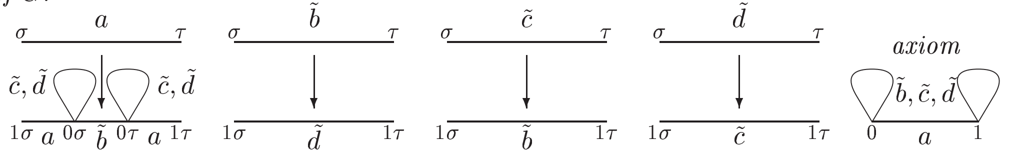
We construct $\mathfrak{G}_{n+1}$ from $\mathfrak{G}_n$. Split Σ^{n+1} as $0\Sigma^n \sqcup 1\Sigma^n$. By virtue of the definition of ϕ given in (37), the b, c, d-edges within $1\Sigma^n$ are in bijection to the c, d, b-edges in $\mathfrak{G}_n$, while the b, c-edges within $0\Sigma^n$ are in bijection with the a-edges in $\mathfrak{G}_n$, and there are d-loops at all $\sigma \in 0\Sigma^n$. Moreover there are "parallel edges" labeled a between 0σ and 1σ for all $\sigma \in \Sigma^n$.

Now consider any b, c, d-edge in $\mathfrak{G}_n$, say between σ and τ. In $\mathfrak{G}_{n+1}$, it gives rise to a d, b, c-edge between 1σ and 1τ.

Consider then an a-edge in $\mathfrak{G}_n$ between σ and τ. In $\mathfrak{G}_{n+1}$, it gives rise to the following subgraph: an a-edge from 1σ to 0σ; two edges, labeled b and c, from 0σ to 0τ; an a-edge from 0τ to 1τ; and two loops, labeled d at 0σ and 0τ. This is precisely the substitutional rule for a, completing the proof. □

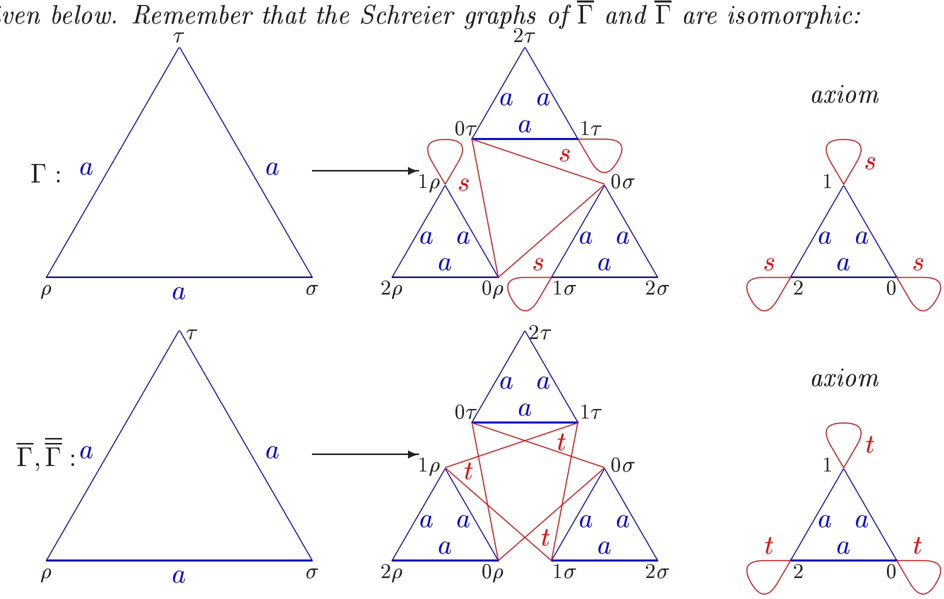
We omit the similar proof of the following result:

THEOREM E.57. *The following four substitutional rules describe the Schreier graph of $\overline{G}$:*



where the vertex inclusions are given by $\sigma \mapsto 1\sigma$ and $\tau \mapsto 1\tau$. The base point is the vertex $111\dots$.

THEOREM E.58. *The substitutional rules producing the Schreier graphs of Γ and $\overline{\Gamma}$ are given below. Remember that the Schreier graphs of $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ are isomorphic:*



where the vertex inclusions are given by $\rho \mapsto 2\rho$, $\sigma \mapsto 2\sigma$ and $\tau \mapsto 2\tau$. The base point is the vertex $222\dots$.

PROOF. We prove the claim for Γ only, as the same reasoning applies for $\overline{\Gamma}$. Consider the Schreier graph $\mathfrak{G}_n$ associated to the action of G on Σ^n, the n-th level of the tree $\mathcal{T}_\Sigma$. The vertex set of $\mathfrak{G}_n$ is Σ^n, and its edges are described by the action of G. Note first that the axiom is $\mathfrak{G}_0$.

We construct $\mathfrak{G}_{n+1}$ from $\mathfrak{G}_n$. Split Σ^{n+1} as $0\Sigma^n \sqcup 1\Sigma^n \sqcup 2\Sigma^n$. By virtue of the definition of ϕ given in (38), the s-edges within $2\Sigma^n$ are in bijection to the s-edges in $\mathfrak{G}_n$, while the s-edges within $0\Sigma^n$ are in bijection with the a-edges in $\mathfrak{G}_n$, and there are s-loops at every $\sigma \in 1\Sigma^n$. Moreover there are "parallel triangles" labeled a between 0σ, 1σ and 2σ for all $\sigma \in \Sigma^n$.

Now consider any s-edge in $\mathfrak{G}_n$, say between σ and τ. In $\mathfrak{G}_{n+1}$, it remains an s-edge, but now between 2σ and 2τ.

Consider then an a-edge in $\mathfrak{G}_n$ between σ and τ. In $\mathfrak{G}_{n+1}$, it gives rise to the following subgraph: an a-triangle between 0σ, 1σ and 2σ; an s-edge between 0σ and 0τ; s-loops at 1σ and 1τ; and an a-triangle between 0τ, 1τ and 2τ. Actually the a-edges form triangles so these subgraphs overlap at the a-triangles and s-loops. This justifies the substitutional rule for a-triangles, completing the proof. □

By Proposition E.12, the limit graphs have asymptotically polynomial growth of degree $\log_2(3)$.

Note that there are maps $\pi_n : V(\mathfrak{G}_{n+1}) \to V(\mathfrak{G}_n)$ that locally (i.e. in each copy of some right-hand rule Y_i) are the inverse of the embedding ι_i. In case these π_n are graph morphisms, and one can consider the projective system $\{\mathfrak{G}_n, \pi_n\}$, and its inverse limit $\widehat{\mathfrak{G}} = \varprojlim \mathfrak{G}_n$, which is a profinite graph [**RZ99**]. We devote our attention to the discrete graph $\mathfrak{G} = \varinjlim \mathfrak{G}_n$.

The growth series of $\mathfrak{G}$ can often be described as an infinite product. We give such an expression for the graph in Figure E.5.2, making use of the fact that $\mathfrak{G}$ "looks like a tree" (even though it is amenable).

Consider the finite graphs $\mathfrak{G}_n$; recall that $\mathfrak{G}_n$ has 3^n vertices. Let D_n be the diameter of $\mathfrak{G}_n$ (maximal distance between two vertices), and let $\gamma_n = \sum_{i\in\mathbb{N}} \gamma_n(i) X^i$ be the growth series of $\mathfrak{G}_n$ (here $\gamma_n(i)$ denotes the number of vertices in $\mathfrak{G}_n$ at distance i from the base point $*$).

The construction rule for $\mathfrak{G}$ implies that $\mathfrak{G}_{n+1}$ can be constructed as follows: take three copies of $\mathfrak{G}_n$, and in each of them mark a vertex V at distance D_n from $*$. At each V delete the loop labeled s, and connect the three copies by a triangle labeled s at the three V's. It then follows that $D_{n+1} = 2D_n + 1$, and $\gamma_{n+1} = (1 + 2X^{D_n+1})\gamma_n$. Using the initial values $\gamma_0 = 1$ and $D_0 = 0$, we obtain by induction

$$D_n = 2^n - 1, \qquad \gamma_n = \prod_{i=0}^{n-1} (1 + 2X^{2^i}).$$

We also have shown that the ball of radius 2^n around $*$ contains 3^n points, so the growth of $\mathfrak{G}$ is at least $n^{\log_2(3)}$. But Proposition E.12 shows that it is also an upper bound, and we conclude:

PROPOSITION E.59. *$\mathfrak{G}$ is an amenable 4-regular graph whose growth function is transcendental, and admits the product decomposition*

$$\gamma(X) = \prod_{i\in\mathbb{N}} (1 + 2X^{2^i}).$$

It is planar, and has polynomial growth of degree $\log_2(3)$.

Any graph is a metric space when one identifies each edge with a disjoint copy of an interval $[0, L]$ for some $L > 0$. We turn $\mathfrak{G}_n$ in a diameter-1 metric space by giving to each edge in $\mathfrak{G}_n$ the length $L = \operatorname{diam}(\mathfrak{G}_n)^{-1}$. The family $\{\mathfrak{G}_n\}$ then converges, in the following sense:

Let A, B be closed subsets of the metric space (X, d). For any ϵ, let $A_\epsilon = \{x \in X \mid d(x, A) \le \epsilon\}$, and define the *Hausdorff distance*

$$d_X(A, B) = \inf\{\epsilon \mid A \subseteq B_\epsilon, B \subseteq A_\epsilon\}.$$

This defines a metric on closed subsets of X. For general metric spaces (A, d) and (B, d), define their *Gromov-Hausdorff distance*

$$d^{GH}(A, B) = \inf_{X,i,j} d_X(i(A), j(B)),$$

where i and j are isometric embeddings of A and B in a metric space X.

We may now rephrase the considerations above as follows: the sequence $\{\mathfrak{G}_n\}$ is convergent in the Gromov-Hausdorff metric. The limit set $\mathfrak{G}_\infty$ is a compact metric space.

The limit spaces are then: for G and $\tilde{G}$, the limit $\mathfrak{G}_\infty$ is the interval $[0,1]$ (in accordance with its linear growth, see Proposition E.12). The limit spaces for Γ, $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ are fractal sets of dimension $\log_2(3)$.

When the present work was completed Stanislav Smirnov informed us about the article [**Mal95**] which has some connection to our paper. A substitutional tree $\mathcal{T}$ is constructed in [**Mal95**], and the spectrum of the Markov operator on $\mathcal{T}$ is of the form $K \sqcup P$, where K is the Julia set of some quadratic map and P is countable. The approximation arguments used in [**Mal95**] are different from ours and do not use any amenability assumption (indeed, the graphs constructed there have eigenvalues with compactly supported eigenvectors).

An essential difference is that the graph in [**Mal95**] is not regular and therefore is not a Schreier graph of a group.

E.6. Concluding Remarks and Problems

Let us draw here some conclusions and formulate a few questions. We hope that our methods will prove useful in the investigation of spectral properties of the Laplace operator Δ for other groups acting on rooted trees. By computing the spectrum of the Hecke type operator of a quasi-regular representation $\rho_{G/P}$ we obtained a subset of the spectrum of Δ; namely, $\operatorname{spec}(\rho_{G/P}) \subseteq \operatorname{spec}(\rho_G)$ as soon as P is amenable. Moreover, in one case (that of $\tilde{G}$) we obtained $\operatorname{spec}(\Delta)$ using the bipartitivity of $\tilde{G}$'s Cayley graph. More research must be done in the nonamenable case, in particular

QUESTION 1. *Under which conditions (besides amenability) does one have* $\operatorname{spec}(\rho_{G/P}) \subseteq \operatorname{spec}(\rho_G)$*? When does one have* $\operatorname{spec}(\rho_{G/P}) = \operatorname{spec}(\rho_G)$*?*

We also hope that the methods in this article can be useful to find residually finite examples of nonamenable groups without free subgroups (all known examples are non-residually-finite). The non-amenability of the group considered could be proven by showing that 1 is an isolated point in the spectrum of the associated dynamical system.

QUESTION 2. *Is there a finitely generated subgroup G of $\mathsf{Aut}(\mathcal{T})$ with no non-abelian free subgroup and such that* $1 \notin \operatorname{spec}(\rho_{G/P})$*?*

Examples of groups G with $1 \notin \operatorname{spec}(\rho_{G/P})$ are also interesting because they provide sequences of expanding graphs [**Lub94**], namely the Schreier graphs $\mathcal{S}(G, P_n, S)$. In some cases these graphs can be Ramanujan graphs. We formulate the following question with Andrzej Żuk:

QUESTION 3. *Are there groups $G < \mathsf{Aut}(\mathcal{T})$, generated by a finite set S of finite automata, such that the sequence of graphs $\mathcal{S}(G, P_n, S)$ is* (i) *a sequence of expanders;* (ii) *a sequence of Ramanujan graphs?*

The problem of factoring the resolvent of operators of Hecke type is related to the decomposition of ρ_{G/P_n} in irreducible representations. It can be shown that any finite-dimensional irreducible representation of G is a subrepresentation in a tensor product of sufficiently many copies of ρ_{G/P_n} (in other words, the representation ring $R(G)$ is generated by the irreducible subrepresentations of ρ_{G/P_n}). We therefore hope that our methods will extend the knowledge on the irreducible representations of G.

We constructed a virtually torsion-free group, Γ, with totally disconnected spectrum $\operatorname{spec}(\rho_{\Gamma/P})$. To answer in the negative to the Baum-Connes conjecture, as well as to the Kadison-Kaplansky conjecture [**Val89**], it would suffice to construct a torsion-free group with a gap in the spectrum of its Laplace operator. Therefore one should delete the "virtually" and replace $\rho_{\Gamma/P}$ by ρ_Γ in order to improve our results.

QUESTION 4. *Is there a torsion-free group $G < \mathsf{Aut}(\mathcal{T})$ with totally disconnected spectrum* $\operatorname{spec}(\rho_G)$*? Or with a gap in its spectrum?*

E.7. Acknowledgments

Both authors wish to thank heartily Pierre de la Harpe for his constant availability and interest. Viviane Baladi, Marc Burger, Vaughan Jones, Volodymyr Nekrashevych, Gilles Robert, Alain Valette and Andrzej Żuk contributed by generous discussions and input.

ANNEXE F

On Parabolic Subgroups and Hecke Algebras of Some Fractal Groups

Laurent Bartholdi and Rostislav I. Grigorchuk

F.1. Introduction

Fractal groups entered recently in the *avant-scène* of group theory, and are related to diverse areas such as the theory of branch groups [**Gri00**], automata groups [**BG99b**] and so on.

Fractal groups of branch type have many interesting properties. Namely, the first examples of groups of intermediate growth were found in this class of groups [**Gri84**]; the simplest examples of infinite finitely-generated torsion groups too [**Gri80a, GS83a**] (thus contributing to the general Burnside problem); fractal groups provide sporadic examples of groups of finite width with unusual associated Lie algebra [**BG00a**], thus answering a question by Efim Zel'manov; etc.

It is therefore of utmost interest to pursue the study of the algebraic, geometric and analytic properties of these groups, and in particular their subgroup structure.

Fractal groups and branch groups are defined in the category of profinite groups as well. These new classes of profinite groups already started to play an important role. For instance, they gave an answer to a question of Efim Zel'manov about groups of finite width [**BG00a**], they were used by Dan Segal [**Seg00**] to solve in the negative a conjecture by Alex Lubotzky, Laci Pyber and Aner Shalev [**Lub95b**] about a gap in the range of subgroup growths, these groups have an universal embedding property [**GHZ98**], and it is believed that branch groups may play an important role in Galois theory [**Bos99**].

Fractal groups are groups acting on regular rooted trees and have self-similarity properties inspired by those of the tree they act on. Branch groups are groups acting on regular (or, more generally, spherically homogeneous [**Gri00**]) rooted trees, and having a *branch structure* that endows them with properties similar to those of the full tree automorphism group.

The action of a fractal group G extends to an action on the boundary of the tree. A *parabolic* subgroup P of G is the stabilizer of an element in the boundary of the tree — or, equivalently, the stabilizer of an infinite geodesical path starting at the root vertex. Parabolic subgroups can be defined for any group acting on a tree, but in the case of branch groups they have the remarkable *weak maximality* property, and the quotient spaces G/P typically have polynomial growth, usually of non-integer degree.

Viewing P as the stabilizer of an infinite path $e = (e_1, e_2, \dots)$, it is approximated by the stabilizers P_n of finite paths $(e_1, \dots, e_n)$, in the sense that $P = \bigcap P_n$. The homogeneous space G/P is then also approximated by the finite spaces G/P_n. These finite spaces have a limit in the Gromov sense, which is a compact finite-dimensional space; in case its Hausdorff dimension is not an integer, we obtain a fractal set of a new nature, as we observed in [**BG99b**]. The study of such spaces is promising.

The present paper contains several new results concerning properties of branch fractal groups, in particular a part of their subgroup lattice and the structure of their parabolic subgroups.

One of the main fruits of this research is the first example of torsion-free branch just-infinite group (see Section F.7). This paper also serves as a companion to [**BG99b**], in that it studies the structure of parabolic subgroups P of fractal groups, and the decomposition of the associated quasi-regular representations $\rho_{G/P}$.

These representations are irreducible, and that there are uncountably many different (pairwise non-equivalent) among them. They are infinite-dimensional, but are approximated by finite-dimensional representations ρ_{G/P_n} where $\{P_n\}$ is a sequence of subgroups of finite index such that $P = \bigcap_{n\in\mathbb{N}} P_n$. For these finite-dimensional representations we describe a decomposition in irreducible components. This decomposition is obtained by a complete description of the structure of the Hecke algebra associated to the pair (G, P). These Hecke algebra turn out to be abelian.

We believe branch fractal groups have good "analytical properties" in the sense that a sufficiently rich representation theory for these groups, their finite images, and the corresponding profinite completions can be developed in order to answer the main questions about harmonic analysis on these groups — their spectrum, the structure of various completions of their group algebra etc.

The set $\{\rho_{G/P} | \, P$ is a parabolic subgroup of $G\}$ is probably sufficiently large for this purpose, since parabolic subgroups have the property that $\bigcap_{g\in G} P^g = 1$ (for the finite-dimensional analogue, this implies that the regular representation ρ_{G_n} is a subrepresentation of the tensor product $\bigotimes_{|\sigma|=n} \rho_{G/\mathsf{Stab}_G(\sigma)}$.)

We are following the first steps along this direction in the present paper. The results given further are already used for the computation of spectra related to fractal groups [**BG99b**], where we show that in some cases these spectra are simple transformations of Julia sets of quadratic maps of the complex plane.

The paper is organized as follows: in Section F.2 we give general definitions concerning groups acting on rooted trees, introduce the congruence property, parabolic subgroups, portraits of elements an Hausdorff dimension of closed subgroups of $\mathsf{Aut}(\mathcal{T})$.

In Section F.3 we recall the definition of branch group, weakly branch group and regular branch group. We prove the weak maximality of parabolic subgroups and provide a criterion evaluating the congruence property for regular branch groups.

In Sections F.4, F.5, F.6, F.7, F.8 we define groups G, $\tilde{G}$, Γ, $\overline{\Gamma}$, $\overline{\overline{\Gamma}}$, and study some properties of these groups. We prove that Γ and $\overline{\Gamma}$ are virtually torsion-free, in contrast to G and $\overline{\overline{\Gamma}}$ which are torsion, and $\tilde{G}$ which is neither torsion nor virtually torsion-free.

We prove that $\tilde{G}$ is (like G [**Gri84**] and Γ, $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ [**Bar00a**]) of intermediate growth, and produce a presentation of $\tilde{G}$ which is of L-type, that is, which involves finitely many relators, along with their iterates under a word substitution. An analogous representation for G was found by Igor Lysionok [**Lys85**].

For each of the involved groups we draw a part of their subgroup lattice, and provide a tree-like decomposition of their parabolic subgroup.

We prove that G, $\tilde{G}$, Γ and $\overline{\overline{\Gamma}}$ are just-infinite branch groups, while $\overline{\Gamma}$ is a just-nonsolvable weakly branch group (the first example of such a group was given in [**BSV99**]).

Finally in Section F.9 we study, for a branch group G, the quasi-regular representations corresponding to parabolic subgroups P and stabilizers P_n of vertices at level n. We show that the quasi-regular representations $\rho_{G/P}$ are irreducible, and for the finite-dimensional representations ρ_{G/P_n} we describe their decomposition in irreducible components, which we explicit in the case of our examples. The Hecke algebra $\mathcal{H}(G, P_n)$, which controls the decomposition of ρ_{G/P_n} in irreducible components, is abelian. As a consequence, the orbit structure of P_n on the homogeneous G-space G/P_n is closely related to that decomposition.

Note that all these results — structure of the parabolic subgroup, lattice of finite-index subgroups, weak maximality of P, abelian Hecke algebra — hold also for the closures $\overline{G}$ of the groups we consider in $\mathsf{Aut}\,\mathcal{T}$, which are branch fractal profinite groups. For instance the statements about the structure of parabolic subgroups are valid for them as well if one replaces the restricted tree-like decomposition by an unrestricted one. Also, in our situation, a group and its closure have the same sequences of (finite) Hecke algebras so one can consider these algebras as associated to profinite groups as well. Four of our groups satisfy the congruence property, so their closures are isomorphic to their profinite completions.

It will be important in the future to develop the theory of representations of profinite branch groups. The results of section F.9 are a first step in this direction. As we obtained a simple description of the double coset decomposition with respect to a parabolic subgroup there is a hope that the classical methods (described for instance in [**CR90**]) as well as more recent developments [**Mic**] will lead to a complete theory of the representations of the considered groups as well as of other groups of this type.

The results in this paper are used in [**BG99b**], and are announced in [**BG00b**].

F.1.1. Notation. The following conventional notations shall be used: for g, h in a group G,

$$g^h = hgh^{-1}, \qquad [g,h] = ghg^{-1}h^{-1};$$

for elements or subsets $g_1, \dots, g_n$ in G, the subgroup they generate is written $\langle g_1, \dots, g_n \rangle$ and its normal closure $\langle g_1, \dots, g_n \rangle^G$.

The symmetric subgroup on a set Σ is written $\mathfrak{S}_\Sigma$.

We also introduce a notation for 'subsemidirect products', as follows:

DEFINITION F.1. *Let A and B be two subgroups of a group G, with $A \cap B = C$, and assume that B is in the normalizer $N_G(A)$ and thus acts on A by conjugation. We write $A \rtimes_C B$ for the subgroup of G generated by A and B, and call it the* subsemidirect product *of A and B.*

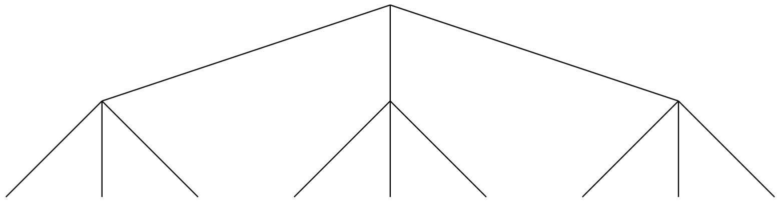
If for some prime p we have $C = \langle B^p, B' \rangle$, then we write $A \rtimes_{p-ab} B$ for the subsemidirect product of A and B, and call it the elementary abelian subsemidirect product *of A and B.*

The motivation for this name is that there is a natural set-bijection between $A \rtimes_C B$ and $(A \times B)/\{(c,c)\}$. The latter need not, however, be a group, since we neither require C to be normal in G nor to centralize A.

F.2. Groups Acting on Rooted Trees

The groups we shall consider will all be subgroups of the group $\mathsf{Aut}(\mathcal{T})$ of automorphisms of a regular rooted tree $\mathcal{T}$. Let Σ be a finite alphabet. The vertex set of the tree $\mathcal{T}_\Sigma$ is the set of finite sequences over Σ; two sequences are connected by an edge when one can be obtained from the other by right-adjunction of a letter in Σ. The top node is the empty sequence $\emptyset$, and the children of σ are all the σs, for $s \in \Sigma$. We shall also consider the *boundary* $\partial\mathcal{T}_\Sigma$ of $\mathcal{T}_\Sigma$ consisting of the semi-infinite sequences over Σ. In most cases we shall write $\mathcal{T}$ for the rooted tree involved, when there is no ambiguity on Σ.

We suppose $\Sigma = \mathbb{Z}/d\mathbb{Z}$, with the operation $\overline{s} = s + 1 \mod d$. Let a, called the *rooted automorphism* of $\mathcal{T}_\Sigma$, be the automorphism of $\mathcal{T}_\Sigma$ defined by $a(s\sigma) = \overline{s}\sigma$: it acts nontrivially on the first symbol only, and geometrically is realized as a cyclic permutation of the d subtrees just below the root.



Fix some Σ, and consider any subgroup $G < \mathsf{Aut}(\mathcal{T})$. Let $\mathsf{Stab}_G(\sigma)$, the *vertex stabilizer* of σ, denote the subgroup of G consisting of the automorphisms that fix the sequence σ, and let $\mathsf{Stab}_G(n)$, the *level stabilizer*, denote the subgroup of G consisting of the automorphisms that fix all sequences of length n:

$$\mathsf{Stab}_G(\sigma) = \{g \in G \mid g\sigma = \sigma\}, \qquad \mathsf{Stab}_G(n) = \bigcap_{\sigma \in \Sigma^n} \mathsf{Stab}_G(\sigma).$$

The $\mathsf{Stab}_G(n)$ are normal subgroups of finite index of G; in particular $\mathsf{Stab}_G(1)$ is of index at most $d!$. Let G_n be the quotient $G/\mathsf{Stab}_G(n)$. If $g \in \mathsf{Aut}(\mathcal{T})$ is an automorphism fixing the sequence σ, we denote by $g_{|\sigma}$ the element of $\mathsf{Aut}(\mathcal{T})$ corresponding to the restriction to sequences starting by σ:

$$\sigma g_{|\sigma}(\tau) = g(\sigma\tau).$$

As the subtree starting from any vertex is isomorphic to the initial tree $\mathcal{T}_\Sigma$, we obtain this way a map

$$\phi : \begin{cases} \mathsf{Stab}_{\mathsf{Aut}(\mathcal{T})}(1) \to \mathsf{Aut}(\mathcal{T})^\Sigma \\ h \mapsto (h_{|0}, \dots, h_{|d-1}) \end{cases} \tag{42}$$

which is an embedding. For a sequence σ and an automorphism $g \in \mathsf{Aut}(\mathcal{T})$, we denote by g^σ the element of $\mathsf{Aut}(\mathcal{T})$ acting as g on the sequences starting by σ, and trivially on the others:

$$g^\sigma(\sigma\tau) = \sigma g(\tau), \qquad g^\sigma(\tau) = \tau \text{ if } \tau \text{ doesn't start by } \sigma.$$

The *rigid stabilizer* of σ is $\mathsf{Rist}_G(\sigma) = \{g^\sigma \mid g \in G\} \cap G$. We also set

$$\mathsf{Rist}_G(n) = \langle \mathsf{Rist}_G(\sigma) \mid \sigma \in \Sigma^n \rangle = \prod_{|\sigma|=n} \mathsf{Rist}_G(\sigma)$$

and call it the *rigid level stabilizer* ($\prod$ denotes direct product). We say G has *infinite rigid stabilizers* if all the $\mathsf{Rist}_G(\sigma)$ are infinite.

DEFINITION F.2. *A subgroup $G < \mathsf{Aut}(\mathcal{T})$ is* spherically transitive *if the action of G on Σ^n is transitive for all $n \in \mathbb{N}$.*

G is fractal *if for every vertex σ of $\mathcal{T}_\Sigma$ one has $\mathsf{Stab}_G(\sigma)_{|\sigma} \cong G$, where the isomorphism is given by identification of $\mathcal{T}_\Sigma$ with its subtree rooted at σ.*

G has the congruence property *if every finite-index subgroup of G contains $\mathsf{Stab}_G(n)$ for some n large enough.*

LEMMA F.3. *The group $G < \mathsf{Aut}(\mathcal{T})$ is fractal if and only if $\phi_{|\mathsf{Stab}_G(1)} : \mathsf{Stab}_G(1) \to \mathsf{Aut}(\mathcal{T})^\Sigma$ is a subdirect embedding into $G \times \cdots \times G$, i.e. if it is an embedding that is surjective on each factor.*

PROOF. If $p_i\phi_{|G} \neq G$ for some projection p_i on the vertex i, then $\mathsf{Stab}_G(i)_{|i} \neq G$ so G is not fractal. We now suppose $\phi_{|G}$ is a subdirect embedding and prove by induction that $\mathsf{Stab}_G(\sigma)_{|\sigma} \cong G$ for all σ.

The induction basis, for $|\sigma| = 1$, is equivalent to the hypothesis. Now by induction $G \to G^{\Sigma^{n-1}}$ is a subdirect embedding, and each factor G maps to G^Σ by $\phi_{|G}$. The composition of two subdirect embeddings is again subdirect, so $G \to G^{\Sigma^n}$ is subdirect. $\square$

For fractal groups, we usually shall write ϕ instead of $\phi_{|G}$.

LEMMA F.4. *A fractal group is spherically transitive if and only if it acts transitively on the first level* Σ.

PROOF. Assume by induction that G is fractal and transitive on Σ^{n-1}, the induction starting at $n=2$. Since ϕ is subdirect, G is transitive on $i\Sigma^{n-1}$ for all $i \in \Sigma$, and since it is transitive on Σ, it is also transitive on Σ^n. □

The full automorphism group $\mathsf{Aut}(\mathcal{T})$ has the structure of a profinite group: it is approximated by the finite groups $\mathsf{Aut}(\mathcal{T})_n = \mathsf{Aut}(\mathcal{T})/\mathsf{Stab}(n)$, and we have

$$\mathsf{Aut}(\mathcal{T}) = \varprojlim_{n\to\infty} \mathsf{Aut}(\mathcal{T})_n.$$

More on the topology of $\mathsf{Aut}(\mathcal{T})$ is said in [**BG99b**]. The following lemma follows directly from the definition of a profinite completion:

LEMMA F.5. *Let $G < \mathsf{Aut}(\mathcal{T})$ have the congruence property. Then its profinite completion $\widehat{G}$ is isomorphic (as a profinite group) to its closure $\overline{G}$ in $\mathsf{Aut}(\mathcal{T})$. If moreover G is contained in a Sylow pro-p-subgroup $\mathsf{Aut}_p(\mathcal{T})$, then $\overline{G}$ is isomorphic to the pro-p completion $\widehat{G}_p$ of G.*

PROOF. By the congruence property, $\{\overline{\mathsf{Stab}_G(n)}\}$ is a basis of neighbourhoods of the identity in $\widehat{G}$. □

DEFINITION F.6. *Assume $G < \mathsf{Aut}(\mathcal{T})$ is given, with a subset $S \subset G$. The* portrait *of $g \in G$ with respect to S is a subtree of $\mathcal{T}$, with inner vertices labeled by $\mathfrak{S}_\Sigma$ and leaf vertices labeled by $S \cup \{1\}$. It is defined recursively as follows: if $g \in S \cup \{1\}$, the portrait of g is the subtree reduced to the root vertex, labeled by g itself. Otherwise, let $\alpha \in \mathfrak{S}_\Sigma$ be the permutation of the top branches of $\mathcal{T}$ such that $g\alpha^{-1} \in \mathsf{Stab}_G(1)$; let $(g_0, \dots, g_{d-1}) = \phi(g\alpha^{-1})$ and let $\mathcal{T}_i$ be the portrait of g_i. Then the portrait of g is the subtree of $\mathcal{T}$ with α labeling the root vertex and subtrees $\mathcal{T}_0, \dots, \mathcal{T}_{d-1}$ connected to the root.*

The portrait *of $g \in G$ is its portrait with respect to $\emptyset$. The element g is called* finitary *if its portrait is finite.*

The depth *of $g \in G$ is the height (length of a maximal path starting at the root vertex) $\partial(g) \in \mathbb{N} \cup \{\infty\}$ of the portrait of g.*

We now suppose $d = p$ is prime, and consider $\mathsf{Aut}_*(\mathcal{T})$, the Sylow pro-$p$ subgroup of $\mathsf{Aut}(\mathcal{T})$ consisting of all elements g whose portrait is labeled by powers of the cycle $(0, 1, \dots, d-1)$. It has the structure of an infinitely iterated wreath product

$$\mathsf{Aut}_*(\mathcal{T}) = \mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}/p\mathbb{Z} \wr \dots.$$

For a closed subgroup G of $\mathsf{Aut}_*(\mathcal{T})$, its *Hausdorff dimension* $\dim_*(G)$ is defined in [**BS97**] as

$$\dim_*(G) = \liminf_{n\to\infty} \frac{p-1}{p^n} \log_p |G_n| = \liminf_{n\to\infty} \frac{\log_p |G_n|}{\log_p |\mathsf{Aut}_*(\mathcal{T})_n|}.$$

In particular, the Hausdorff dimension of $\mathsf{Aut}_*(\mathcal{T})$ is 1, and $\dim_*$ is invariant upon taking finite-index subgroups.

F.3. Branch Groups

We consider now a special class of groups acting on rooted trees. We shall always implicitly assume they act spherically transitively.

DEFINITION F.7. (1) *G is a* regular branch *group if it has a finite-index normal subgroup $K < \mathsf{Stab}_G(1)$ such that*

$$K^\Sigma < \phi(K).$$

It is then said to be regular branch over K.

(2) *A subgroup $G < \mathsf{Aut}(\mathcal{T})$ is a* branch group *if for every $n \geq 1$ the subgroup $\mathsf{Rist}_G(n)$ has finite index in G.*
(3) *G is a* weak branch *group if all of its rigid stabilizers $\mathsf{Rist}_G(\sigma)$ are infinite.*

Note that the definition of a branch group admits an even more general setting — see [**Gri00**]. Four of our examples will be regular branch groups, and the last one will not be a branch, but rather a weak branch group. The following lemma shows that, for fractal groups, 1 implies 2 implies 3 in Definition F.7.

LEMMA F.8. *If G is a fractal, regular branch group, then it is a branch group. If G is a branch group, then it is a weak branch group.*

PROOF. Assume G is a regular branch group over its subgroup K. Clearly K^{Σ^n} can be viewed, through ϕ^n, as a subgroup of $\mathsf{Rist}_G(n)$, and is of finite index in G^{Σ^n}, so $\mathsf{Rist}_G(n)$ is of finite index in G. The second implication holds because branch groups are infinite, and 'finite index in an infinite group' is stronger than 'infinite'. □

If G is a regular branch group over its subgroup K, the following notations is also introduced: given a subgroup L of K, we write $L_{(0)} = L$ and inductively $L_{(n)} = \phi^{-1}(L^{\Sigma}_{(n-1)})$. These $L_{(i)}$ form a sequence of subgroups of L with $\bigcap_{n\geq 0} L_{(n)} = \{1\}$.

DEFINITION F.9. *A group G is* just-infinite *if it is infinite, and for any non-trivial normal subgroup N the quotient G/N is finite.*

Note that in checking just-infiniteness one may restrict one's attention to subgroups $N = \langle g\rangle^G$, i.e. normal closures of a non-trivial element of G. We will use the following criterion:

PROPOSITION F.10. *Let G be regular branch over K. Then G is just infinite if and only if $|K : K'| < \infty$.*

PROOF. Clearly if K' is of infinite index in K then $\langle K'\rangle^G$ is of infinite index in G, and is not trivial (K clearly cannot be abelian) so G is not just infinite.

Conversely, assume $|K : K'| < \infty$ and let $G \ni g \neq 1$. Let $N = \langle g\rangle^G$; we will show that N is of finite index. Determine n such that $g \in \mathsf{Stab}_{\mathsf{Aut}(\mathcal{T})}(n-1) \setminus \mathsf{Stab}_{\mathsf{Aut}(\mathcal{T})}(n)$. Then there is a sequence σ of length $n-1$ such that $g_{|\sigma} \notin \mathsf{Stab}_{\mathsf{Aut}(\mathcal{T})}(1)$. Choose now two elements k_1, k_2 of K. Because G is branched on K, it contains for $i = 1, 2$ the elements $\xi_i = k_i^{\sigma 0}$. Let $\eta = [\xi_1, g] \in N$. It fixes all sequences except: those starting by $\sigma 0$, upon which it acts as k_1, and possibly those starting by σx for $x \geq 1$. Consider $\zeta = [\eta, \xi_2] \in N$. Clearly $\zeta = [k_0, k_1]^{\sigma 0}$; as the commutator $[k_0, k_1]$ was chosen arbitrarily, it follows that N contains $K'^{\sigma 0}$; and as N is normal, it contains $K' \times \cdots \times K'$, the product having d^n factors. Now K' is of finite index in K which is of finite index in G, so $K' \times \cdots \times K'$ is of finite index in G and the same holds for N. □

DEFINITION F.11. *A group G is* just-non-solvable *if it is not solvable, but all its non-trivial quotients are solvable.*

PROPOSITION F.12. *Let G be regular branch over K. Then G is just-non-solvable if and only if $G/K_{(1)}$ is solvable. In particular, if d is prime, every regular branch subgroup of $\mathsf{Aut}_*(\mathcal{T})$ is just-non-solvable.*

PROOF. Let G be just-non-solvable. Then $K_{(1)}$ is a non-trivial normal subgroup, so $G/K_{(1)}$ is solvable.

Let now N be a non-trivial normal subgroup of G. It is shown in [**Gri00**, Theorem 4] that N contains $K'_{(n)}$ for some $n \in \mathbb{N}$, so it suffices to show that all $G/K'_{(n)}$ are solvable. Consider the chain

$$G/K'_{(n)} \triangleright K/K'_{(n)} \triangleright \cdots \triangleright K_{(n)}/K'_{(n)} \cong K/K' \times \cdots \times K/K'.$$

The last group is abelian, hence solvable ; successive quotients in the sequence are also solvable because

$$(K_{(i)}/K'_{(n)})/(K_{(i+1)}/K'_{(n)}) = K_{(i)}/K_{(i+1)} \cong K/K_{(1)} \times \cdots \times K/K_{(1)},$$

and $K/K_{(1)}$ is solvable by assumption. Also, $(G/K'_{(n)})/(K/K'_{(n)}) \cong G/K$ is solvable; therefore $G/K'_{(n)}$ is an extension of solvable groups, so is solvable.

In case d is prime and G is a branch subgroup of $\mathsf{Aut}_*(\mathcal{T})$, the quotient $G/K_{(1)}$ is a finite d-group, so is solvable. □

The following criterion describes which branch groups enjoy the congruence property:

Proposition F.13. *Let G be regular branch over K. Then G has the congruence property if and only if K contains $\mathsf{Stab}_G(m)$ for some $m \in \mathbb{N}$.*

Proof. Let N be a finite-index subgroup of G. By replacing N with its core $\bigcap_{g\in G} N^g$, still of finite index, we may suppose N is normal in G. In the quotient G/N, consider the descending sequence $\{K_{(n)}/N\}$. Since $\bigcap_{n\in\mathbb{N}} K_{(n)} = \{1\}$ and G/N is finite, there is an $n \in \mathbb{N}$ such that $K_{(n)}/N = 1$, whence $K_{(n)} < N$. We then have

$$N > K_{(n)} > \mathsf{Stab}_G(m)_{(n)} > \mathsf{Stab}_G(m+n).$$

□

When furthermore G is finitely generated, the following 'quantitative congruence property' shall be useful to prove equalities among subgroups:

Proposition F.14 (Quantitative Congruence Property). *Let G be regular branch over K, finitely generated by the set S and with the congruence property. Let n be minimal such that $\langle s\rangle^G \geq \mathsf{Stab}_G(n)$ for all generators $s \in S$. Let m be minimal such that K' contains $\mathsf{Stab}_G(m)$.*

Let N be any non-trivial normal subgroup of G and $1 \neq g \in N$. Then N contains $\mathsf{Stab}_G(\partial(g) + m + n)$.

Proof. This follows again from Theorem 4 in [**Gri00**]. □

F.3.1. Parabolic subgroups. In the context of groups acting on the hyperbolic space, a parabolic subgroup is the stabilizer of a point on the boundary. We give here a few general facts concerning parabolic subgroups of fractal or branched groups, and recall some results on growth of groups and sets on which they act.

Definition F.15. *Let $\mathcal{T} = \Sigma^*$ be a rooted tree. A* ray *e in $\mathcal{T}$ is an infinite geodesic starting at the root of $\mathcal{T}$, or equivalently an element of $\partial\mathcal{T} = \Sigma^{\mathbb{N}}$.*

Let $G < \mathsf{Aut}(\mathcal{T})$ be any subgroup and e be a ray. The associated parabolic subgroup *is $P_e = \mathsf{Stab}_G(e)$.*

The following important facts are easy to prove:

- $\bigcap_{e\in\partial\mathcal{T}} P_e = \bigcap_{g\in G} P^g = 1$.
- Let e be an infinite ray and define the subgroups $P_n = \mathsf{Stab}_G(e_1 \dots e_n)$. Then the P_n have index d^n in G (since G acts transitively) and satisfy
$$P_e = \bigcap_{n\in\mathbb{N}} P_n.$$
- P has infinite index in G, and has the same image as P_n in the quotient $G_n = G/\mathsf{Stab}_G(n)$.

Definition F.16. *Let G be a group generated by a finite set S, let X be a set upon which G acts transitively, and choose $x \in X$. The* growth *of X is the function $\gamma : \mathbb{N} \to \mathbb{N}$ defined by*

$$\gamma(n) = |\{gx \in X \mid |g| \leq n\}|,$$

where $|g|$ denotes the minimal length of g when written as a word over S. By the growth *of G we mean the growth of the action of G on itself by left-multiplication.*

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there is a constant $C \in \mathbb{N}$ such that $f(n) < Cg(Cn + C) + C$ for all $n \in \mathbb{N}$, and $f \sim g$ if $f \preceq g$ and $g \preceq f$. The equivalence class of the growth of X is independent of the choice of S and of x.

X is of polynomial growth *if $\gamma(n) \preceq n^d$ for some d. It is of* exponential growth *if $\gamma(n) \succeq e^n$. It is of* intermediate growth *in the remaining cases. This trichotomy does not depend on the choice of x or S.*

DEFINITION F.17. *Two infinite sequences $\sigma, \tau : \mathbb{N} \to \Sigma$ are* confinal *if there is an $N \in \mathbb{N}$ such that $\sigma_n = \tau_n$ for all $n \geq N$.*

Confinality is an equivalence relation, and equivalence classes are called confinality classes.

The following result is due to Volodymyr Nekrashevych and Vitaly Sushchansky.

PROPOSITION F.18. *Let G be a group acting on a regular rooted tree $\mathcal{T}$, and assume that for any generator $g \in G$ and infinite sequence σ, the sequences σ and $g\sigma$ differ only in finitely many places. Then the confinality classes of the action of G on $\partial\mathcal{T}$ are unions of orbits. If moreover $\mathsf{Stab}_G(\sigma)$ contains the rooted automorphism a for all $\sigma \in \mathcal{T}$, the orbits of the action are confinality classes.*

The proof of the following result appears in [**BG99b**].

PROPOSITION F.19. *Let $G < \mathsf{Aut}(\mathcal{T})$ satisfy the conditions of Proposition F.18, and suppose that for the map $\phi : g \mapsto (g_1, \dots, g_d)$ defined in (42) there are constants λ, μ such that $|g_i| \leq \lambda|g| + \mu$ for all i. Let P be a parabolic subgroup. Then G/P, as a G-set, is of polynomial growth of degree at most $\log_{1/\lambda}(d)$. If moreover G is spherically transitive, then G/P's asymptotical growth is polynomial of degree $\log_{1/\lambda'}(d)$, with λ' the infimum of the λ as above.*

DEFINITION F.20. *Let G be a branch group, and H any subgroup. We say H is* weakly maximal *if H is of infinite index in G, but all subgroups of G strictly containing H are of finite index in G.*

Note that every infinite finitely generated group admits maximal subgroups, by Zorn's lemma.

Note also that some branch groups may not contain any infinite-index maximal subgroup; this is the case for G, as was shown by Ekaterina Pervova.

PROPOSITION F.21. *Let P be a parabolic subgroup of a regular branch group G. Then P is weakly maximal.*

PROOF. Assume G is regular branch over K, and $P = \mathsf{Stab}_G(e)$. Recall that G contains a product of d^n copies of K at level n, and clearly P contains a product of $d^n - 1$ copies of K at level n, namely all but the one indexed by the vertex $e_1 \dots e_n$.

Take $g \in G \setminus P$. There is then an $n \in \mathbb{N}$ such that $g(e_n) \neq e_n$, so $\langle P, P^g \rangle$ contains the product of d^n copies of K at level n, hence is of finite index in G. □

F.4. The Group G

We give here the basic facts we will use about the first of Grigorchuk's examples, the group G [**Gri80a, Har00**]. We apply the discussion of the previous section to $\Sigma = \{0, 1\}$. Recall a is the automorphism permuting the top two branches of $\mathcal{T}_2$. Define recursively by b the automorphism acting as a on the right branch and c on the left, by c the automorphism acting as a on the right branch and d on the left, and by d the automorphism acting as 1 on the right branch and b on the left. In formulæ,

$$\begin{aligned} b(0x\sigma) &= 0\overline{x}\sigma, & b(1\sigma) &= 1c(\sigma), \\ c(0x\sigma) &= 0\overline{x}\sigma, & c(1\sigma) &= 1d(\sigma), \\ d(0x\sigma) &= 0x\sigma, & d(1\sigma) &= 1b(\sigma). \end{aligned}$$

G is the group generated by $\{a, b, c, d\}$. It is readily checked that these generators are of order 2 and that $\{1, b, c, d\}$ constitute a Klein group; one of the generators $\{b, c, d\}$ can thus be omitted.

We write $H_n = \mathsf{Stab}_G(n)$ and $H = H_1$. Explicitly, the map ϕ restricts to

$$\phi : \begin{cases} H \to G \times G \\ b \mapsto (a, c), \quad b^a \mapsto (c, a) \\ c \mapsto (a, d), \quad c^a \mapsto (d, a) \\ d \mapsto (1, b), \quad d^a \mapsto (b, 1). \end{cases}$$

Define also the following subgroups of G:

$$B = \langle b \rangle^G = \langle b, b^a, b^{da}, b^{ada} \rangle,$$
$$K = \langle (ab)^2 \rangle^G = \langle (ab)^2, (abad)^2, (adab)^2 \rangle.$$

The group G

- is an infinite torsion 2-group.
- is of intermediate growth.
- is amenable.
- is fractal and branched on its subgroup K.
- is just-infinite.
- is residually finite.
- has an infinite recursive presentation [**Lys85**] of L-type
$$G = \langle a, b, c, d |\, a^2, b^2, c^2, d^2, bcd, \sigma^i(ad)^4, \sigma^i(adacac)^4 \ (i \in \mathbb{N}) \rangle,$$
where σ is the substitution on $\{a, b, c, d\}^*$ defined by
$$\sigma(a) = aca, \quad \sigma(b) = d, \sigma(c) = b, \sigma(d) = c,$$
which induces an injective expanding endomorphism of G of infinite-index image. Moreover none of the relators of G are superfluous [**Gri99**].
- The subgroup B is of index 8 in G and K is of index 16. Also, K contains $\mathsf{Stab}_G(3)$ and K' contains $K_{(2)}$, so G has the congruence property.
- The quotients $G_n = G/\mathsf{Stab}_G(n)$ have order $2^{5 \cdot 2^{n-3}+2}$ for $n \geq 3$ (and order 2^{2^n-1} for $n \leq 3$). Therefore the closure $\overline{G}$ of G in $\mathsf{Aut}(\mathcal{T})$ is isomorphic to the profinite completion $\widehat{G}$ and is a pro-2-group. It has Hausdorff dimension $5/8$ [**Gri00**].

Most of these facts are known, and appear in the extensive reference [**Har00**] and in [**Gri00**]. One can then check by direct computation that K is of index 16. To prove that G is regular branch, set $x = (ab)^2$. Then one has $\phi([x, d]) = (x, 1)$ so by conjugation $\phi(K) > K \times 1$ and thus $\phi(K) > K \times K$. By direct computation, K' is of index 64 in K, so G is just-infinite.

For all other computations, we propose an alternate method of proof, based on the following

LEMMA F.22. *G satisfies the Quantitative Congruence Property for $m = 4$ and $n = 3$.*

PROOF. This follows from the above description of K. □

PROPOSITION F.23. *We have*

$$\phi(H) = (B \times B) \rtimes \langle \phi(c), \phi(c^a) \rangle,$$
$$\phi(B) = (K \times K) \rtimes_{\langle \phi(ab)^8 \rangle} \langle \phi(b), \phi(b^a) \rangle,$$
$$\phi(K) = (K \times K) \rtimes_{\langle \phi(ab)^8 \rangle} \langle \phi(ab)^2 \rangle,$$

with the notation introduced in Definition F.1.

PROOF. Each of these subgroups H, B, K are normal finite-index subgroups of G. By the Quantitative Congruence Property, they are all contained in some $\mathsf{Stab}_G(n)$. It is therefore equivalent to study them directly or to study their images in $G_n = G/\mathsf{Stab}_G(n)$, which is a finite group. A computer algebra system, like [**S**$^+$**93**], can then be used to derive their structure. ☐

F.4.1. Low-Index subgroups. G has 7 subgroups of index 2:

$$\langle b, ac\rangle, \qquad \langle c, ad\rangle, \quad \langle d, ab\rangle,$$
$$\langle b, a, a^c\rangle, \qquad \langle c, a, a^d\rangle, \langle d, a, a^b\rangle,$$
$$H = \langle b, c, b^a, c^a\rangle.$$

As can be computed from its presentation [**Lys85**] and a computer algebra system [**S**$^+$**93**], G has the following subgroup count:

Index	Subgroups	Normal	In H	Normal
1	1	1	0	0
2	7	7	1	1
4	19	7	9	4
8	61	7	41	7
16	237	5	169	5
32	843	3	609	3

F.4.2. Normal closures of generators. They are as follows:

$$A = \langle a\rangle^G = \langle a, a^b, a^c, a^d\rangle, \qquad G/A \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$
$$B = \langle b\rangle^G = \langle b, b^a, b^{ad}, b^{ada}\rangle, \qquad G/B \cong D_8,$$
$$C = \langle c\rangle^G = \langle c, c^a, c^{ad}, c^{ada}\rangle, \qquad G/C \cong D_8,$$
$$D = \langle d\rangle^G = \langle d, d^a, d^{ac}, d^{aca}\rangle, \qquad G/D \cong D_{16}.$$

F.4.3. Some other subgroups. To complete the picture, we introduce the following subgroups of G:

$$K = \langle (ab)^2\rangle^G, \qquad L = \langle (ac)^2\rangle^G, \qquad M = \langle (ad)^2\rangle^G,$$
$$\overline{B} = \langle B, L\rangle, \qquad \overline{C} = \langle C, K\rangle, \qquad \overline{D} = \langle D, K\rangle,$$
$$T = K^2 = \langle (ab)^4\rangle^G,$$
$$T_{(m)} = \underbrace{T \times \cdots \times T}_{2^m}, \qquad K_{(m)} = \underbrace{K \times \cdots \times K}_{2^m}, \qquad N_{(m)} = T_{(m-1)}K_{(m)}.$$

THEOREM F.24.

- *In the Lower Central Series,* $\gamma_{2^m+1}(G) = N_{(m)}$ *for all* $m \geq 1$.
- *In the Derived Series,* $K^{(n)} = \mathsf{Rist}_G(2n)$ *for all* $n \geq 2$ *and* $G^{(n)} = \mathsf{Rist}_G(2n-3)$ *for all* $n \geq 3$.
- *The rigid stabilizers satisfy*

$$\mathsf{Rist}_G(n) = \begin{cases} D & \text{if } n = 1, \\ K_{(n)} & \text{if } n \geq 2. \end{cases}$$

- *The level stabilizers satisfy*

$$\mathsf{Stab}_G(n) = \begin{cases} H & \text{if } n = 1, \\ \langle D, T\rangle & \text{if } n = 2, \\ \langle N_{(2)}, (ab)^4(adabac)^2\rangle & \text{if } n = 3, \\ \underbrace{\mathsf{Stab}_G(3) \times \cdots \times \mathsf{Stab}_G(3)}_{2^{n-3}} & \text{if } n \geq 4. \end{cases}$$

- *There is for all $\sigma \in \Sigma^n$ a surjection $\cdot_{|\sigma} : \mathsf{Stab}_G(n) \twoheadrightarrow G$ given by projection on the factor indexed by σ.*

The top of the lattice of normal subgroups of G below H is given in Table 1.

Proof. The first three points are proven by Alexander Rozhkov in [**Roz96b**]. To prove the fourth assertion, we apply Lemma F.22 to determine the structure of $\mathsf{Stab}_G(n)$ for $n \le 4$, and note that $\mathsf{Stab}_G(4) = \mathsf{Stab}_G(3) \times \mathsf{Stab}_G(3)$.

For the last statement, note that $s = (ab)^4(adabac)^2$ belongs to $\mathsf{Stab}_G(3)$), and that $\phi^n(\sigma^{n-3}(s)) = (1, \dots, 1, ba, d, d, ba, a, c, a, c)$ giving, after conjugation, any generator of G in any position on any level n. □

F.4.4. The Subgroup P. Let e be the ray 1^∞ and let P be the corresponding parabolic subgroup.

Theorem F.25. *P/P' is an infinite elementary 2-group generated by the images of c, $d = (1, b)$ and of all elements of the form $(1, \dots, 1, (ac)^4)$ in $\mathsf{Rist}_G(n)$ for $n \in \mathbb{N}$. The following decomposition holds:*

$$P = \Big(B \times \Big(\big(K \times ((K \times \dots) \rtimes \langle (ac)^4 \rangle)\big) \rtimes \langle b, (ac)^4 \rangle\Big)\Big) \rtimes \langle c, (ac)^4 \rangle,$$

where each factor (of nesting n) in the decomposition acts on the subtree just below some e_n but not containing e_{n+1}.

Note that we use the same notation for a subgroup B or K acting on a subtree, keeping in mind the identification of a subtree with the original tree. The same convention will hold for Theorems F.41, F.50, F.60, F.67, and all related propositions. Note also that ϕ is omitted when it would make the notations too heavy.

Proof. Define the following subgroups of G_n:

$$B_n = \langle b \rangle^{G_n}; \qquad K_{(n)} = \langle (ab)^2 \rangle^{G_n};$$
$$Q_n = B_n \cap P_n; \qquad R_n = K_{(n)} \cap P_n.$$

Then the theorem follows from the following proposition. □

Proposition F.26. *These subgroups have the following structure:*

$$P_n = (B_{n-1} \times Q_{n-1}) \rtimes \langle c, (ac)^4 \rangle;$$
$$Q_n = (K_{n-1} \times R_{n-1}) \rtimes \langle b, (ac)^4 \rangle;$$
$$R_n = (K_{n-1} \times R_{n-1}) \rtimes \langle (ac)^4 \rangle.$$

Proof. A priori, P_n, as a subgroup of H_n, maps in $(B_{n-1} \times B_{n-1}) \rtimes \langle (a, d), (d, a) \rangle$. Restricting to those pairs that fix e_n gives the result. Similarly, Q_n, as a subgroup of B_n, maps in $(K_{n-1} \times K_{n-1}) \rtimes \langle (a, c), (c, a) \rangle$, and R_n, as a subgroup of K_n, maps in $(K_{n-1} \times K_{n-1}) \rtimes \langle (ac, ca), (ca, ac) \rangle$. □

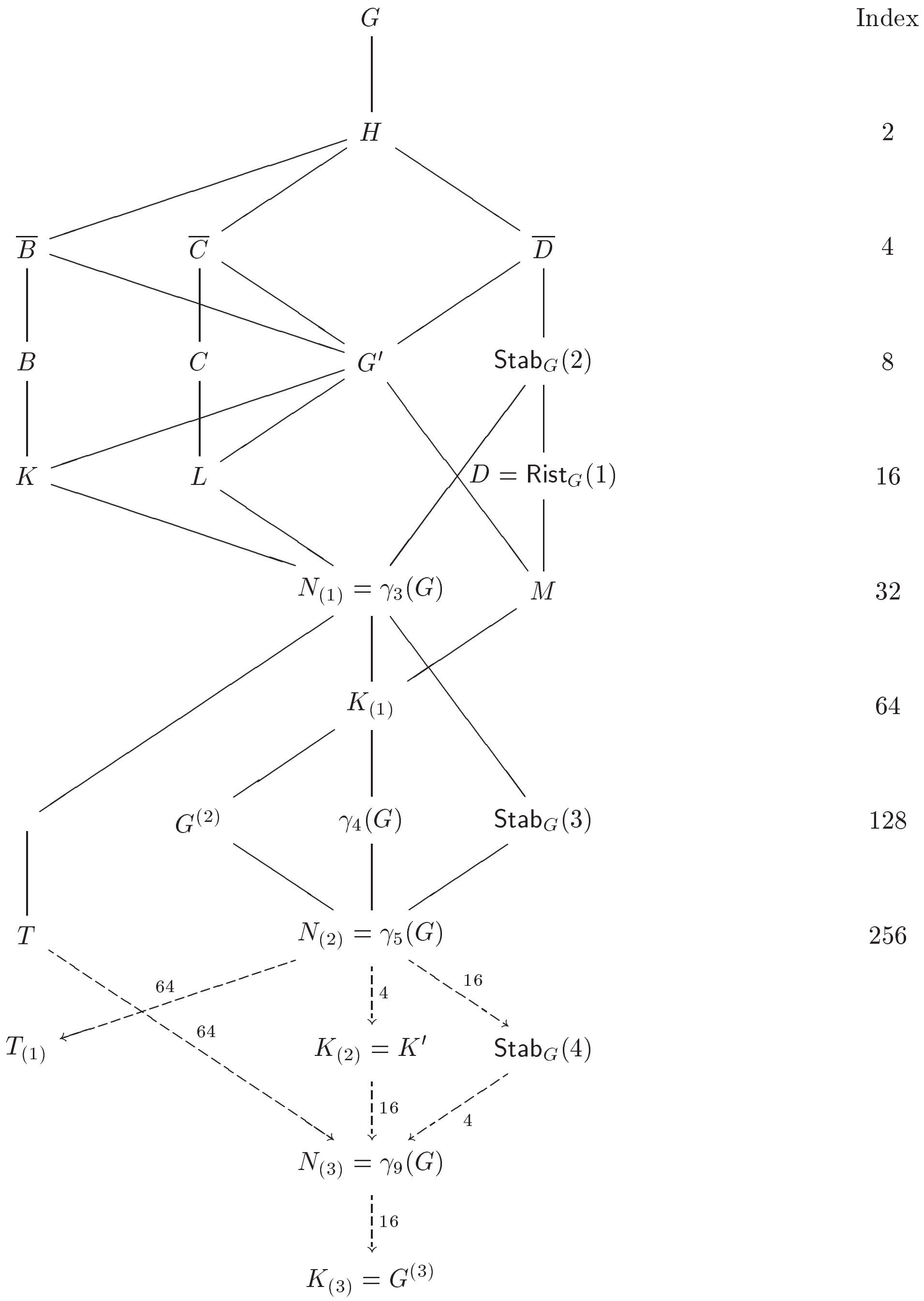
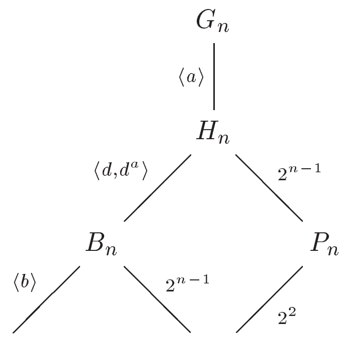


Table 1: The top of the lattice of normal subgroups of G below H. The index of the inclusions are indicated next to the edges.

Corollary F.27. *The group G_n and its subgroups $H_n, B_n, K_n, P_n, R_n, Q_n$ are arranged in a lattice*



where the quotients or the indices are represented next to the arrows.

F.5. The Group $\tilde G$

We describe another fractal group, acting on the same tree $\mathcal{T}_2$ as G. We denote again by a the automorphism permuting the top two branches, and recursively by $\tilde b$ the automorphism acting as a on the right branch and $\tilde c$ on the left, by $\tilde c$ the automorphism acting as 1 on the right branch and $\tilde d$ on the left, and by $\tilde d$ the automorphism acting as 1 on the right branch and $\tilde b$ on the left. In formulæ,

$$\begin{aligned} \tilde b(0x\sigma) &= 0\overline{x}\sigma, & \tilde b(1\sigma) &= 1\tilde c(\sigma),\\ \tilde c(0\sigma) &= 0\sigma, & \tilde c(1\sigma) &= 1\tilde d(\sigma),\\ \tilde d(0\sigma) &= 0\sigma, & \tilde d(1\sigma) &= 1\tilde b(\sigma). \end{aligned}$$

Then $\tilde G$ is the group generated by $\{a, \tilde b, \tilde c, \tilde d\}$. Clearly all these generators are of order 2, and $\{\tilde b, \tilde c, \tilde d\}$ is elementary abelian of order 8.

We write $\tilde H_n = \mathsf{Stab}_{\tilde G}(n)$ and $\tilde H = \tilde H_1$. Explicitly, the map ϕ restricts to

$$\phi : \begin{cases} \tilde H \to \tilde H \times \tilde H \\ \tilde b \mapsto (a, \tilde c), \quad \tilde b^a \mapsto (\tilde c, a) \\ \tilde c \mapsto (1, \tilde d), \quad \tilde c^a \mapsto (d, 1) \\ \tilde d \mapsto (1, \tilde b), \quad \tilde d^a \mapsto (\tilde b, 1). \end{cases}$$

Proposition F.28. *$\tilde G$ contains elements of finite and infinite order.*

Proof. Consider the element $x = a\tilde b\tilde c\tilde d$ of $\tilde G$. Then $x^2 \in \tilde H$ satisfies $\phi(x^2) = (x, x)$, so $x^{2^n} \neq 1$ for all n; as $\tilde G < \mathsf{Aut}(\mathcal{T}_2)$ has only 2-torsion, it follows that x is of infinite order. □

Note that x acts on $\partial\mathcal{T}_2$ like an 'adding machine' (see [**BORT96**]). More generally, every spherically transitive automorphism of $\mathcal{T}_p$ is conjugated in $\mathsf{Aut}(\mathcal{T}_p)$ to a standard one, called the *adding machine*, that can be written $z \mapsto z+1$ after identification of $\partial\mathcal{T}_p$ with $\mathbb{Z}_p$.

Proposition F.29. *$\tilde G$ contains G as a subgroup of infinite index.*

Proof. The embedding is given by $a \mapsto a$, $b \mapsto \tilde b\tilde d$, $c \mapsto \tilde c\tilde b$, $d \mapsto \tilde d\tilde c$. The index is infinite because the subgroups G and $\langle a\tilde b\tilde c\tilde d\rangle$ do not intersect, one being torsion and the other torsion-free. □

Define the elements $u = (a\tilde b)^2$ and $v = (a\tilde d)^2$ in $\tilde G$, and consider its following subgroups:

$$\tilde H = \langle \tilde b, \tilde c, \tilde d\rangle^{\tilde G}, \qquad \tilde B = \langle \tilde b, \tilde d\rangle^{\tilde G}, \qquad \tilde C = \langle \tilde b, v\rangle^{\tilde G}, \qquad \tilde K = \langle u, v\rangle^{\tilde G}.$$

Proposition F.30. *They have the following structure:*

$\tilde H = \langle \tilde b, \tilde c, \tilde d, \tilde b^a, \tilde c^a, \tilde d^a\rangle$ *is normal of index* 2 *in* $\tilde G$.

$\tilde B = \langle \tilde b, \tilde d, \tilde b^a, \tilde d^a, \tilde b^{\tilde c a}, \tilde b^{a\tilde c a}\rangle$ *is normal of index* 8 *in* $\tilde G$.

$\tilde C = \langle \tilde b, v, \tilde b^a, \tilde b^{\tilde c a}, \tilde b^{a\tilde c a}\rangle$ *is normal of index* 16 *in* $\tilde G$.

$\tilde K = \langle u, v, (a\tilde b\tilde d)^2, u^a, u^{a\tilde c}\rangle$ *is normal of index* 32 *in* $\tilde G$.

Furthermore,

$$\begin{aligned}
\phi(\tilde H) &= (\tilde B \times \tilde B) \rtimes \langle \phi(\tilde b), \phi(\tilde b^a)\rangle,\\
\phi(\tilde B) &= (\tilde C \times \tilde C) \rtimes \langle \phi(\tilde b), \phi(\tilde b^a)\rangle,\\
\phi(\tilde C) &= (\tilde K \times \tilde K) \rtimes_{\phi\langle\langle[\tilde b,v],[\tilde b^a,v]\rangle} \langle \phi(\tilde b), \phi(\tilde b^a), v\rangle,\\
\phi(\tilde K) &= (\tilde K \times \tilde K) \rtimes_{\phi\langle[u,v]\rangle^{\tilde G}} \langle u, v\rangle.
\end{aligned}$$

PROPOSITION F.31. *$\tilde G$ is spherically transitive, fractal and regular branch over its subgroup $\tilde K$.*

PROOF. $\tilde G$ is fractal by Lemma F.3 and the nature of the map ϕ. As $\tilde K$ is normal, $\phi(\tilde K)$ contains $\phi[u,\tilde d] = (1,u)$ and $\phi[u,\tilde c] = (1,v)$, so by conjugation it contains $1 \times \tilde K$ and $\tilde K \times 1$, so finally it contains $\tilde K \times \tilde K$. □

PROPOSITION F.32. *$\tilde G$ is just-infinite.*

PROOF. By direct computation, $[\tilde K : \tilde K'] = 64$. Apply Proposition F.10. □

PROPOSITION F.33. *Define the substitution $\tilde\sigma$ on $\{a, \tilde b, \tilde c, \tilde d\}^*$ by*

$$\tilde\sigma : \begin{cases} a \mapsto a\tilde b a, & \tilde b \mapsto \tilde d,\\ \tilde c \mapsto \tilde b, & \tilde d \mapsto \tilde c. \end{cases}$$

Then $\tilde G$ has a recursive presentation of L-type

$$(43)\quad \tilde G = \Big\langle a, \tilde b, \tilde c, \tilde d \Big| a^2, \tilde b^2, \tilde c^2, \tilde d^2, [\tilde b, \tilde c], [\tilde b, \tilde d], [\tilde c, \tilde d],$$
$$\tilde\sigma^i(a\tilde c)^4, \tilde\sigma^i(a\tilde d)^4, \tilde\sigma^i(a\tilde c a\tilde d)^2, \tilde\sigma^i(a\tilde b)^8, \tilde\sigma^i(a\tilde b a\tilde b a\tilde c)^4, \tilde\sigma^i(a\tilde b a\tilde b a\tilde d)^4, \tilde\sigma^i(a\tilde b a\tilde b a\tilde c a\tilde b a\tilde b a\tilde d)^2 \quad (i \ge 0)\Big\rangle,$$

and $\tilde\sigma$ induces an injective expanding endomorphism of $\tilde G$ of infinite-index image.

PROOF. Consider the groups

$$\begin{aligned}
\Gamma &= \langle \alpha, \beta, \gamma, \delta | \alpha^2, \beta^2, \gamma^2, \delta^2, [\beta, \gamma], [\beta, \delta], [\gamma, \delta], (\alpha\gamma)^4\rangle,\\
\Xi &= \langle \beta, \gamma, \delta, \beta^\alpha, \gamma^\alpha, \delta^\alpha\rangle <_2 \Gamma.
\end{aligned}$$

Then $\tilde G$ is a quotient of Γ, written $\tilde G = \Gamma/\Omega$, via the map $\alpha \mapsto a, \beta \mapsto b, \gamma \mapsto c, \delta \mapsto d$, and the map ϕ lifts to a map $\theta : \Xi \to \Gamma \times \Gamma$. Define

$$\Omega_n = \{g \in \Gamma | \theta^n \text{ is applicable and } \theta^n(g) = (1, \dots, 1) \quad (2^n \text{ copies})\},$$

where the notation implies that $g \in \Xi, \theta(g) \in \Xi \times \Xi, \dots$. For any word w in $\{\alpha, \beta, \gamma, \delta\}$ of length at least 2 representing an element of Ξ, the corresponding words $\theta(w)_{1,2}$ will be strictly shorter; thus every $g \in \Omega$ eventually gives 1 through iterated application of θ, and thus $\Omega = \cup_{n\ge 0}\Omega_n$. We will obtain an explicit set of generators for Ω_n: let $\omega_0 = (\alpha\gamma)^4$ and

$$\{\omega_1, \dots, \omega_6\} = \{(\alpha\delta)^4, (\alpha\gamma\alpha\delta)^2, (\alpha\beta)^8, (\alpha\beta\alpha\beta\alpha\gamma)^4, (\alpha\beta\alpha\beta\alpha\delta)^4, (\alpha\beta\alpha\beta\alpha\gamma\alpha\beta\alpha\beta\alpha\delta)^2\}.$$

Then we claim that for all $n \ge 0$

$$\Omega_n = \langle \tilde\sigma^{j+1}(\omega_0), \tilde\sigma^j(\omega_i) \quad (0 \le j \le n-1, 1 \le i \le 6)\rangle^\Gamma.$$

By direct application of the Todd-Coxeter algorithm [**S**$^+$**93**], we obtain the presentation

$$\Xi = \Big\langle \beta, \gamma, \delta, \overline\beta, \overline\gamma, \overline\delta \Big| \beta^2, \gamma^2, \delta^2, \overline\beta^2, \overline\gamma^2, \overline\delta^2, [\beta, \gamma], [\beta, \delta], [\gamma, \delta], [\overline\beta, \overline\gamma], [\overline\beta, \overline\delta], [\overline\gamma, \overline\delta], [\gamma, \overline\gamma]\Big\rangle.$$

Computation shows that $\theta(\Xi)$ is of index 8 in $\Gamma\times\Gamma$. From this we obtain, again using Todd-Coxeter, the presentation

$$\begin{aligned}\theta(\Xi)=\Big\langle\beta,\gamma,\delta,\overline{\beta},\overline{\gamma},\overline{\delta}\,\Big|\,&\beta^2,\gamma^2,\delta^2,\overline{\beta}^2,\overline{\gamma}^2,\overline{\delta}^2,[\beta,\gamma],[\beta,\delta],[\gamma,\delta],[\overline{\beta},\overline{\gamma}],[\overline{\beta},\overline{\delta}],[\overline{\gamma},\overline{\delta}],\\ &[\gamma,\overline{\gamma}],[\gamma,\overline{\delta}],[\delta,\overline{\gamma}],[\delta,\overline{\delta}],(\beta\overline{\beta})^4,(\beta\overline{\beta}x\beta\overline{\beta}\overline{y})^2\ (x,y\in\{\gamma,\delta\})\Big\rangle.\end{aligned}$$

As a consequence, we can write $\ker(\theta)$ as a normal subgroup of Γ by keeping only those relators of $\theta(\Xi)$ that do not appear in Ξ and rewriting them in $\{\alpha,\beta,\gamma,\delta\}$, namely

$$\Omega_1=\ker(\theta)=\langle\omega_1,\dots,\omega_6\rangle^\Gamma.$$

Then a direct computation shows that $\theta\tilde\sigma(\omega_i)=(1,\omega_i)$ for $i=0,\dots,6$. This proves that

$$\begin{aligned}\Omega_n&=\{g\in\Xi|\,\theta(g)\in\Omega_{n-1}\times\Omega_{n-1}\}\\ &=\big(\{\tilde\sigma(g)|\,g\in\Omega_{n-1}\}\cup\Omega_{n-1}\big)^\Gamma\\ &=\langle\tilde\sigma^{j+1}(\omega_0),\tilde\sigma^j(\omega_i)\quad(0\le j\le n-1,1\le i\le 6)\rangle^\Gamma.\end{aligned}$$

□

Corollary F.34. *All relations of $\tilde G$ have even length. As a consequence, the Cayley graph of $\tilde G$ relative to the generating set $\{a,\tilde b,\tilde c,\tilde d\}$ is bipartite.*

We believe the relations given in the previous theorem are independent, and that the method used in [**Gri99**] can be used to prove this.

Note that the relations of G can be obtained from those of $\tilde G$; in the following equalities we indicate by an underscore the letters affected by a relation in $\tilde G$.

$$\begin{aligned}(ad)^4&=(a\tilde c\tilde d)^4=_{\tilde G}(a\tilde c\tilde d a\tilde d\tilde c)^2=_{\tilde G}(a\tilde c\tilde d a\tilde d(\tilde d a)^4\tilde c)(a\tilde c\tilde d a\tilde d\tilde c)\\ &=_{\tilde G}(\underline{a\tilde c a\tilde d}\ \underline{a\tilde d a\tilde c})(a\tilde d\tilde c a\tilde c\tilde d)=_{\tilde G}(\tilde d a\tilde c a\tilde c a\tilde d a)(a\tilde d\tilde c a\tilde c\tilde d)=_{\tilde G}\tilde d(a\tilde c)^4\tilde d=_{\tilde G}\tilde d^2=_{\tilde G}1,\end{aligned}$$

and

$$\begin{aligned}(adacac)^4&=(a\tilde d\underline{\tilde c a\tilde c}\tilde b a\tilde b\tilde c)^4=_{\tilde G}(\underline{a\tilde d a\tilde c}a\tilde c a\tilde b a\tilde b\tilde c)^4=_{\tilde G}\tilde c a(\tilde d\underline{\tilde c a\tilde b a\tilde b a})(\tilde d\tilde c a\tilde b a\tilde b a)^3a\tilde c\\ &=_{\tilde G}\tilde c a\big(\tilde d(a\tilde b a\tilde b a\tilde c)^3(\tilde d\tilde c a\tilde b a\tilde b a)^3\big)a\tilde c\\ &=_{\tilde G}\tilde c a\big(\tilde d(a\tilde b a\tilde b a\tilde c)\underline{(a\tilde b a\tilde b a\tilde c)(a\tilde b a\tilde b a\tilde d a\tilde b a\tilde b a)(\tilde c}\tilde d a\tilde b a\tilde b a)(\tilde d\tilde c a\tilde b a\tilde b a)\big)a\tilde c\\ &=_{\tilde G}\tilde c a\big(\tilde d(a\tilde b a\tilde b a\tilde c)\underline{(\tilde d a\tilde b a\tilde b a)^2(\tilde d}\tilde c a\tilde b a\tilde b a)\big)a\tilde c=_{\tilde G}\tilde c a\tilde d(a\tilde b a\tilde b a\tilde c a\tilde b a\tilde b a\tilde d)^2\tilde d a\tilde c=_{\tilde G}1.\end{aligned}$$

Proposition F.35. *The finite quotients $\tilde G_n=\tilde G/\operatorname{\mathsf{Stab}}_{\tilde G}(n)$ of $\tilde G$ have order $2^{13\cdot 2^{n-4}+2}$ for $n\ge 4$, and 2^{2^n-1} for $n\le 4$.*

Proof. For $n\ge 4$, $\phi(\tilde H)$ is a subgroup of index 8 in $\tilde G\times\tilde G$, so $\tilde G_n$ is a subgroup of index 8 in $\tilde G_{n-1}\wr\mathbb{Z}/2$ and $|\tilde G_n|=|\tilde G_{n-1}|^2/4$. For $n\le 4$ one has $\tilde G_n=\operatorname{\mathsf{Aut}}(\mathcal{T})_n=\mathbb{Z}/2\wr\cdots\wr\mathbb{Z}/2$. □

Proposition F.36. *$\tilde K\ge\operatorname{\mathsf{Stab}}_{\tilde G}(4)$ and $\tilde K'\ge\tilde K'_{(2)}$, so $\tilde G$ has the congruence property. Additionally, $\tilde K'\ge\operatorname{\mathsf{Stab}}_{\tilde G}(5)$.*

Proof. The first and third assertions can be checked on a computer. For the second, K contains $y=[u,d]$ and $z=[u,c]$; these elements satisfy $\phi(y)=(1,u)$ and $\phi(z)=(1,v)$. Then K' contains $[y,v]=\phi^{-2}(1,1,u,1)$ and $[z,d]=\phi^{-2}(1,1,v,1)$, so it contains $\phi^{-2}(1\times 1\times K\times 1)$ and $K_{(2)}$. □

Corollary F.37. *The closure $\overline{\tilde G}$ of $\tilde G$ in $\operatorname{\mathsf{Aut}}(\mathcal{T})$ is isomorphic to the profinite completion $\widehat{\tilde G}$ and is a pro-2-group. It has Hausdorff dimension $13/16$.*

F.5.1. The Growth of $\tilde{G}$. By the growth of a group one means the growth, in the sense of Definition F.16, of the group acting on itself. We rephrase the definition of growth of a group in a slightly more general frame:

DEFINITION F.38. *Let G be a group generated by a finite set S, and let $\nu : S \to \mathbb{R}_+^*$ be any function. The* weight *of $g \in G$ is*

$$|g| = \min\{\nu(s_1) + \cdots + \nu(s_n) \,|\, s_1 \cdots s_n = g, s_i \in S\}.$$

The growth series *of G with respect to ν is*

$$F_\nu(\tau) = \sum_{g \in G} e^{\tau |g|}.$$

This series converges at least in the half-plane $\Re(\tau) < -\log(n)/\min_{s\in S} \nu(s)$. Let $\rho(\nu)$, the growth rate *of G with respect to ν, be the smallest non-positive value such that the series converges.*

PROPOSITION F.39. *If $\rho(\nu) < 0$, then G has exponential growth, while if $\rho(\nu) = 0$, then G has intermediate or polynomial growth.*

PROOF. Let m and M be the minimum and maximum of the weight function ν, and set $R = \lim \sqrt[n]{\gamma_G^S(n)}$. By considering the series $F_S(\tau) = \sum_{n\geq 0} \gamma(n)\tau^n$, whose radius of convergence is $1/R$ and comparing it with $F_\nu(\tau)$, we obtain

$$M\rho(\nu) \leq \log(1/R) \leq m\rho(\nu),$$

so $R > 1$ is equivalent to $\rho(\nu) < 0$. □

The first examples of groups of intermediate growth were constructed in [**Gri83**]; the group G is one of them.

THEOREM F.40. *$\tilde{G}$ has intermediate growth.*

PROOF. First, note that $\tilde{G}$ cannot have polynomial growth, since it contains G whose growth function is greater than $e^{\sqrt{n}}$ [**Bar00d**].

Take as generators for $\tilde{G}$ the set $S = \{a, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{b}\tilde{c}, \tilde{b}\tilde{d}, \tilde{c}\tilde{d}, \tilde{b}\tilde{c}\tilde{d}\}$; let θ be strictly between the real root of the equation $-2 + X + X^2 + X^3 = 0$ and 1, for instance $\theta = 0.811$ and let ν be defined by

$$\begin{aligned}
\nu(a) &= 1,\\
\nu(\tilde{b}) &= (\theta + \theta^3)/(1 - \theta^3) \approx 2.87,\\
\nu(\tilde{c}) &= (-1 + \theta + \theta^2 + \theta^3)/(1 - \theta^3) \approx 2.14,\\
\nu(\tilde{d}) &= (\theta^2 + \theta^3)/(1 - \theta^3) \approx 2.54,\\
\nu(\tilde{b}\tilde{c}) &= (-1 + \theta + \theta^3)/(1 - \theta^3) \approx 0.73,\\
\nu(\tilde{b}\tilde{d}) &= (\theta^3)/(1 - \theta^3) \approx 1.13,\\
\nu(\tilde{c}\tilde{d}) &= (-1 + \theta^2 + \theta^3)/(1 - \theta^3) \approx 0.41,\\
\nu(\tilde{b}\tilde{c}\tilde{d}) &= (1 + \theta^3)/(1 - \theta^3) \approx 3.28.
\end{aligned}$$

Clearly any element $g \in G$, when expressed as a minimal word in S, will have the form $[a]x_1 a x_2 \ldots a x_n [a]$, where the first and last a are optional and $x_i \in S \setminus \{a\}$. Indeed the function ν satisfies the triangular inequalities $\nu(\tilde{b}) + \nu(\tilde{c}) < \nu(\tilde{b}\tilde{c})$, etc. Choose once and for all a minimal expression for every element of $\tilde{G}$.

Suppose now for contradiction that $\rho(\nu) < 0$. For some value $\eta \in (0, 1)$ to be chosen later, partition $\tilde{G}$ in two sets: A containing those elements $g \in G$ whose minimal expression $s_1 \dots s_n$ contains at least ηn occurrences of the generator $x = \tilde{b}\tilde{c}\tilde{d}$, and B the other elements. Define two generating series

$$F_A(\tau) = \sum_{g \in A} e^{\tau |g|}, \qquad F_B(\tau) = \sum_{g \in A} e^{\tau |g|}.$$

Clearly $F_\nu = F_A + F_B$. We will show that for an appropriate value of η both F_A and F_B will converge up to some σ with $\rho(\nu) < \sigma < 0$.

We bound F_A by replacing A by a larger set, namely the set of all words $s_1 \dots s_n$ containing at least ηn occurrences of x. Then

$$F_A(\tau) < \sum_{n \geq 0} \binom{n}{\eta n} \left(\sum_{s \in S} e^{\tau \nu(s)} \right)^{(1-\eta)n} \left(e^{\tau \nu(x)} \right)^{\eta n}.$$

By Stirling's formula,

$$\binom{n}{\eta n} \approx \frac{\sqrt{2\pi\eta(1-\eta)}\sqrt{n}}{(\eta^\eta + (1-\eta)^{1-\eta})^n}.$$

Putting these together, we conclude that F_A converges up to any $\sigma > \rho(\nu)$ if

$$\frac{\left(\sum_{s \in S} e^{\sigma \nu(s)} \right)^{1-\eta} \left(e^{\sigma \nu(x)} \right)^{\eta}}{\eta^\eta (1-\eta)^{1-\eta}} < 1,$$

and this will hold for η large enough, as both the first multiplicand and the denominator tend to 1 as η tends to 1, while the second multiplicand tends to $e^{\sigma \nu(x)} < 1$.

We then approximate F_B by considering the subset $B' \subset B$ of words $s_1 \dots s_n$ that either start or end by a, but not both; and further that contain an even number of as. The series $F_{B'}(\tau)$ obtained this way will satisfy $F_B \approx 4F_{B'}$. Now B' injects in $G \times G$ through the map ϕ, written $g \mapsto (g_{|0}, g_{|1})$. We will compare $|g|$ with $|g_{|0}| + |g_{|1}|$. Thanks to the choice of ν, every generator $s \neq x$ contributing $\nu(s)$ to $|g|$ will contribute at most $\theta\nu(s)$ to $|g_{|0}| + |g_{|1}|$, while every x contributing $\nu(x)$ to $|g|$ will contribute $\nu(x)$ to $|g_{|0}| + |g_{|1}|$. We conclude that for all $g \in B'$ we have

$$\frac{|g_{|0}| + |g_{|1}|}{|g|} < \frac{\eta\nu(x) + (1-\eta)\min\nu}{\eta\nu(x) + (1-\eta)\theta\min\nu} =: \zeta < 1.$$

This means every element of weight n in B' can be written as a pair of elements of G with total weight at most ζn, or in formulæ

$$F_{B'}(\tau) \leq (F_\nu(\zeta\tau))^2.$$

The series $F_{B'}$ thus converges up to $\zeta\rho(\nu) > \rho(\nu)$; the same holds for F_B. Then the series F_ν converges up to $\min(\zeta\rho(\nu), \sigma) > \rho(\nu)$, a contradiction. □

F.5.2. The Subgroup $\tilde{P}$. Let e be the ray 1^∞ and let $\tilde{P}$ be the corresponding parabolic subgroup.

THEOREM F.41. *$\tilde{P}/\tilde{P}'$ is an infinite elementary 2-group generated by the images of $\tilde{b}$, $\tilde{c} = (1, \tilde{d})$, $\tilde{d} = (1, \tilde{b})$ and of all elements of the form $(1, \dots, 1, u^2)$ or $(1, \dots, 1, v)$. The following decomposition holds:*

$$\tilde{P} = \left(\tilde{B} \times \left(\left(\tilde{C} \times ((\tilde{K} \times \dots) \rtimes \langle u^2, v \rangle) \right) \rtimes \langle \tilde{b}, u^2, \tilde{d}, v \rangle \right) \right) \rtimes \langle \tilde{b}, u^2 \rangle.$$

Define the following subgroups of $\tilde{G}_n$:

$$\tilde{B}_n = \langle \tilde{b}, \tilde{d} \rangle^{\tilde{G}_n}; \qquad \tilde{C}_n = \langle \tilde{b}, \tilde{v} \rangle^{\tilde{G}_n}; \qquad \tilde{K}_n = \langle u, v \rangle^{\tilde{G}_n};$$
$$\tilde{Q}_n = \tilde{B}_n \cap \tilde{P}_n; \qquad \tilde{R}_n = \tilde{C}_n \cap \tilde{P}_n; \qquad S_n = \tilde{K}_n \cap \tilde{P}_n.$$

PROPOSITION F.42. *These subgroups have the following structure:*

$$\begin{aligned}
&\tilde{P}_n = (\tilde{B}_{n-1} \times \tilde{Q}_{n-1}) \rtimes \langle \tilde{b}, u^2 \rangle;\\
&\tilde{Q}_n = (\tilde{C}_{n-1} \times \tilde{R}_{n-1}) \rtimes \langle \tilde{b}, u^2 \rangle;\\
&\tilde{R}_n = (\tilde{K}_{n-1} \times \tilde{S}_{n-1}) \rtimes_{\langle [b,v] \rangle} \langle b, u^2, v \rangle . \tilde{S}_n \qquad = (\tilde{K}_{n-1} \times \tilde{S}_{n-1}) \rtimes_{\langle [u^2,v] \rangle} \langle u^2, v \rangle.
\end{aligned}$$

PROOF. The claims match those of Proposition F.30, and are proved by restricting to elements preserving e_n the 'y' and 'z' in decompositions of the kind $(x \times y) \rtimes z$. $\square$

COROLLARY F.43. *The group $\tilde{G}_n$ and its subgroups $\tilde{H}_n, \tilde{B}_n, \tilde{C}_n, \tilde{K}_n, \tilde{P}_n, \tilde{R}_n, \tilde{Q}_n, \tilde{S}_n$ are arranged in a lattice*

$$\begin{tikzcd}
 & & & \tilde{G}_n \arrow[d, dash, "\langle a \rangle"'] & \\
 & & & \tilde{H}_n \arrow[dl, dash, "\langle \tilde{c}, \tilde{c}^a \rangle"'] \arrow[dr, dash, "2^{n-1}"] & \\
 & & \tilde{B}_n \arrow[dl, dash, "\langle \tilde{d} \rangle"'] \arrow[dr, dash, "2^{n-1}"] & & \tilde{P}_n \arrow[dl, dash, "2^2"] \\
 & \tilde{C}_n \arrow[dl, dash, "\langle \tilde{b} \rangle"'] \arrow[dr, dash, "2^{n-1}"] & & \tilde{Q}_n \arrow[dl, dash, "2"] & \\
 K_n \arrow[dr, dash, "2^{n-1}"'] & & R_n \arrow[dl, dash, "2"] & & \\
 & \tilde{S}_n & & &
\end{tikzcd}$$

where the quotients or the indices are represented next to the arrows.

F.6. The Group Γ

The next three groups we study are subgroups of $\mathsf{Aut}(\mathcal{T}_3)$. Denote by a the automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let t be the automorphisms of $\mathcal{T}_3$ defined recursively by

$$t(0x\sigma) = 0\overline{x}\sigma, \qquad t(1x\sigma) = 1x\sigma, \qquad t(2\sigma) = 2t(\sigma).$$

Then Γ is the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, t\}$; its growth was studied by Jacek Fabrykowski and Narain Gupta [**FG91**].

We write $H_n = \mathsf{Stab}_\Gamma(n)$ and $H = H_1$. Explicitly, the map ϕ restricts to

$$\phi : \Big\{ t \to (a, 1, t), \quad t^a \to (t, a, 1), \quad t^{a^2} \to (1, t, a).$$

Define the elements $x = at$, $y = ta$ of Γ. Let K be the subgroup of Γ generated by x and y, and let L be the subgroup of K generated by K' and cubes in K.

Proposition F.44. *We have the following diagram of normal subgroups:*

$$\begin{tikzcd}
 & \Gamma \arrow[dl, dash, "\langle a|\, a^3\rangle"'] \arrow[dr, dash, "\langle a|\, a^3\rangle"] & \\
K \arrow[dr, dash, "\langle x|\, x^3\rangle"'] & & H=\mathsf{Stab}_\Gamma(1) \arrow[dl, dash, "\langle t|\, t^3\rangle"] \\
 & \Gamma'=K\cap H=[K,H] \arrow[d, dash] & \\
 & L=\langle K',K^3\rangle=\gamma_3(\Gamma) \arrow[dl, dash] \arrow[dr, dash] & \\
K' \arrow[d, dash] & & H'=\phi^{-1}(\Gamma'\times\Gamma'\times\Gamma')=\mathsf{Stab}_\Gamma(2) \\
\langle L\times L\times L, x^3y^{-3}, [x,y^3]\rangle=\gamma_4(\Gamma) \arrow[d, dash] & & \\
\langle L\times L\times L, [x,y^3]\rangle=\gamma_5(\Gamma) & &
\end{tikzcd}$$

where the quotients are represented next to the arrows; all edges represent normal inclusions of index 3*. Furthermore* $L = K \cap \phi^{-1}(K \times K \times K)$.

Proof. First we prove K is normal in Γ, of index 3, by writing $y^t = x^{-1}y^{-1}$, $y^{a^{-1}} = y^{-1}x^{-1}$, $y^{t^{-1}} = y^a = x$; similar relations hold for conjugates of x. A transversal of K in Γ is $\langle a\rangle$. All subgroups in the diagram are then normal.

Since $[a,t] = y^{-1}x = t^a t^{-1}$, we clearly have $\Gamma' < K\cap H$. Now as $\Gamma' \neq K$ and $\Gamma' \neq H$ and Γ' has index 3^2, we must have $\Gamma' = K \cap H$. Finally $[a,t] = [x,t]^{t^{-1}}$, so $\Gamma' = [K,H]$.

Next $x^3 = [a,t][t,a^{-1}][a^{-1},t^{-1}]$ and similarly for y, so $K^3 < \Gamma'$ and $L < \Gamma'$. Also, $\phi[x,y] = (y^{-1},y^{-1},x^{-1})$ and $\phi x^3 = (y,x,y)$ both belong to $K\times K\times K$, while $[a,t]$ does not; so L is a proper subgroup of Γ', of index 3 (since K/L is the elementary abelian group $(\mathbb{Z}/3\mathbb{Z})^2$ on x and y).

Consider now H'. It is in $\mathsf{Stab}_\Gamma(2)$ since $H = \mathsf{Stab}_\Gamma(1)$. Also, $[t,t^a] = y^3[y^{-1},x]$ and similarly for other conjugates of t, so $H' < L$, and $\phi[t,t^a] = ([a,t],1,1)$, so $\phi(H') =$

$\Gamma' \times \Gamma' \times \Gamma'$. Finally H' it is of index 3 in L (since $H/H' = (\mathbb{Z}/3\mathbb{Z})^3$ on $t, t^a, t^{a^{-1}}$), and since $\mathsf{Stab}_\Gamma(2)$ is of index 3^4 in Γ (with quotient $\mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}/3\mathbb{Z}$) we have all the claimed equalities. □

Proposition F.45. *Γ is a just-infinite fractal group, is regular branch over Γ', and has the congruence property.*

Proof. Γ is fractal by Lemma F.3 and the nature of the map ϕ. By direct computation, $[\Gamma : \Gamma'] = [\Gamma' : \phi^{-1}(\Gamma' \times \Gamma' \times \Gamma')] = [\phi^{-1}(\Gamma' \times \Gamma' \times \Gamma') : \Gamma''] = 3^2$, so Γ is branched on Γ'. Then $\Gamma'' = \gamma_5(\Gamma)$, as is shown in [**Bar00c**], so Γ'' has finite index and Γ is just-infinite by Proposition F.10.

$\Gamma' \geq \mathsf{Stab}_\Gamma(2)$, so Γ has the congruence property. □

Proposition F.46. *We have, with the notation introduced in Definition F.1,*

$$\phi(H) = (\Gamma' \times \Gamma' \times \Gamma') \rtimes_{3-ab} \langle t, t^a, t^{a^2} \rangle,$$
$$\phi(\Gamma') = (\Gamma' \times \Gamma' \times \Gamma') \rtimes_{3-ab} \langle [a,t], [a^2,t] \rangle.$$

Theorem F.47. *The subgroup K of Γ is torsion-free; thus Γ is virtually torsion-free.*

Proof. For $1 \neq g \in K$, let $|g|_t$, the t-length of g, denote the minimal number of $t^{\pm 1}$'s required to write g as a word over the alphabet $\{a^{\pm 1}, t^{\pm 1}\}$. We will show by induction on $|g|_t$ that g is of infinite order.

First, if $|g|_t = 1$, i.e. $g \in \{x^{\pm 1}, y^{\pm 1}\}$, we conclude from $\phi(x^3) = (*, *, x)$ and $\phi(y^3) = (*, *, y)$ that g is of infinite order.

Suppose now that $|g|_t > 1$. If $g \in L$, then $\phi(g) = (g_0, g_1, g_2) \in K \times K \times K$ and it suffices to show that one of the g_i is of infinite order—this follows by induction since $|g_i|_t < |g|_t$ and some $g_i \neq 1$. We may thus suppose that $g \in K \setminus L$. Up to symmetry, it suffices also to consider elements g of the form ℓx, ℓxy and ℓxy^{-1}, for $\ell \in L$. Write $\phi(\ell) = (\ell_0, \ell_1, \ell_2)$.

In the first case, we have $\phi(g^3) = \phi(\ell x)^3 = (a\ell_2 t \ell_1 \ell_0, *, *)$. It suffices to show that the first coordinate of this expression is non-trivial, as K contains at worst only 3-torsion, being contained in the 3-Sylow of $\mathsf{Aut}(\mathcal{T}_3)$. Now map Γ to Γ/Γ', an elementary abelian group of order 9. One checks that $\ell_0\ell_1\ell_2 \equiv 1$ in the abelian quotient, so the first coordinate maps to $\overline{at} \not\equiv 1$ in Γ/Γ'.

The second case is handled in the same way. Finally, if $g = \ell xy^{-1}$, then $\phi(g^3) \in L \times L \times L$, so $\phi^2(g^3) \in K \times \cdots \times K$ (9 factors); each factor has strictly smaller t-length than g, and as before the projection in one of the coordinates onto the abelian quotient gives some $x \not\equiv 1$. □

Proposition F.48. *The finite quotients $\Gamma_n = \Gamma/H_n$ of Γ have order $3^{3^{n-1}+1}$ for $n \geq 2$, and 3 for $n = 1$.*

Proof. Follows immediately from $[\Gamma : \Gamma'] = 3^2$ and $[\Gamma' : \phi^{-1}(\Gamma' \times \Gamma' \times \Gamma')] = 3^2$. □

Corollary F.49. *The closure $\overline{\Gamma}$ of Γ in $\mathsf{Aut}(\mathcal{T})$ is isomorphic to the profinite completion $\widehat{\Gamma}$ and is a pro-3-group. It has Hausdorff dimension $1/3$.*

F.6.1. The Subgroup P. Let e be the infinite sequence 2^∞, and let P be the corresponding parabolic subgroup.

Theorem F.50. *P/P' is an infinite elementary 3-group generated by t, t^a and all elements of the form $(1, \dots, 1, [a,t])$. The following decomposition holds:*

$$P = \Bigg(\Gamma' \times \Gamma' \times \Big(\big(\Gamma' \times \Gamma' \times ((\Gamma' \times \Gamma' \times \dots) \rtimes_{3-ab} \langle [a,t] \rangle)\big) \rtimes_{3-ab} \langle [a,t] \rangle\Big)\Bigg) \rtimes_{3-ab} \langle t, t^a \rangle,$$

where each factor (of nesting n) in the decomposition acts on the subtree just below some e_n but not containing e_{n+1}.

Define the following subgroups of Γ_n:

$$\Gamma'_n = \langle [a,t]\rangle^{\Gamma_n}; \qquad Q_n = \Gamma'_n \cap P_n.$$

PROPOSITION F.51. *These subgroups have the following structure:*

$$P_n = (\Gamma'_{n-1} \times \Gamma'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle t, t^a\rangle;$$
$$Q_n = (\Gamma'_{n-1} \times \Gamma'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle [a,t]\rangle.$$

F.7. The Group $\overline{\Gamma}$

Recall a denotes the automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let now t be the automorphism of $\mathcal{T}_3$ defined recursively by

$$t(0x\sigma) = 0\overline{x}\sigma, \qquad t(1x\sigma) = 1\overline{x}\sigma, \qquad t(2\sigma) = 2t(\sigma).$$

Then $\overline{\Gamma}$ is the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a,t\}$.

We write $H_n = \mathsf{Stab}_{\overline{\Gamma}}(n)$ and $H = H_1$. Explicitly, the map ϕ restricts to

$$\phi : \Big\{ t \to (a,a,t), \quad t^a \to (t,a,a), \quad t^{a^2} \to (a,t,a).$$

Define the elements $x = ta^{-1}$, $y = a^{-1}t$ of $\overline{\Gamma}$, and let K be the subgroup of $\overline{\Gamma}$ generated by x and y. Then K is normal in $\overline{\Gamma}$, because $x^t = y^{-1}x^{-1}$, $x^a = x^{-1}y^{-1}$, $x^{t^{-1}} = x^{a^{-1}} = y$, and similar relations hold for conjugates of y. Moreover K is of index 3 in $\overline{\Gamma}$, with transversal $\langle a\rangle$.

LEMMA F.52. *H and K are normal subgroups of index 3 in $\overline{\Gamma}$, and $\overline{\Gamma}' = \mathsf{Stab}_K(1) = H \cap K$ is of index 9; furthermore $\phi(H\cap K) \triangleleft K \times K \times K$. For any element $g = (u,v,w) \in \phi(H \cap K)$ one has $wvu \in H \cap K$.*

PROOF. First note that $\mathsf{Stab}_K(1) = \langle x^3, y^3, xy^{-1}, y^{-1}x\rangle$, for every word in x and y whose number of a's is divisible by 3 can be written in these generators. Then compute

$$\phi(x^3) = (y, x^{-1}y^{-1}, x), \qquad \phi(y^3) = (x^{-1}y^{-1}, x, y),$$
$$\phi(xy^{-1}) = (1, x^{-1}, x), \qquad \phi(y^{-1}x) = (y, 1, y^{-1}).$$

The last assertion is also checked on this computation. $\square$

Proposition F.53. *Writing $c = [a,t] = x^{-1}y^{-1}x^{-1}$ and $d = [x,y]$, we have the following diagram of normal subgroups:*

$$\begin{tikzcd}
& \overline{\Gamma} \arrow[dl, dash, "\langle a|\, a^3\rangle"'] \arrow[dr, dash, "\langle a|\, a^3\rangle"] & \\
K \arrow[dr, dash, "\langle x,y|\, x^3,y^3,x=y\rangle"'] & & H \arrow[dl, dash, "\langle t_0,t_1,t_2|\, t_0^3,t_0=t_1=t_2\rangle"] \\
& \overline{\Gamma}' = \langle c, c^t, c^{a^{-1}}, c^{at}\rangle = K \cap H = [K,H] \arrow[dl, dash, "\mathbb{Z}^2"'] \arrow[dr, dash, "(\mathbb{Z}/3\mathbb{Z})^2"] & \\
K' = \langle d, d^t, d^{a^{-1}}, d^{at}\rangle \arrow[dr, dash, "\mathbb{Z}^2"'] & & H' \arrow[dl, dash] \\
& \overline{\Gamma}'' = \phi^{-1}(K' \times K' \times K') \arrow[d, dash, "(\mathbb{Z}/3\mathbb{Z})^2"] & \\
& K'' &
\end{tikzcd}$$

where the quotients are represented next to the arrows; additionally,

$$K/K' = \langle x,y|\,[x,y]\rangle \cong \mathbb{Z}^2,$$
$$\overline{\Gamma}'/\overline{\Gamma}'' = \langle c, c^t, c^{a^{-1}}, c^{at}|\,[c,c^t],\dots\rangle \cong \mathbb{Z}^4,$$
$$K'/K'' = \langle d, d^t, d^{a^{-1}}, d^{at}|\,[d,d^t],\dots,(d/d^{at})^3,(d^{a^{-1}}/d^t)^3\rangle \cong \mathbb{Z}^2 \times (\mathbb{Z}/3\mathbb{Z})^2.$$

Writing each subgroup in the generators of the groups above it, we have

$$K = \langle x = at^{-1}, y = a^{-1}t\rangle,$$
$$H = \langle t, t_1 = t^a, t_2 = t^{a^{-1}}\rangle,$$
$$\overline{\Gamma}' = \langle b_1 = xy^{-1}, b_2 = y^{-1}x, b_3 = x^3, b_4 = y^3\rangle = \langle c_1 = tt_1^{-1}, c_2 = tt_1t, c_3 = tt_2^{-1}, c_4 = tt_2t\rangle.$$

Corollary F.54. *The congruence property does not hold for $\overline{\Gamma}$; nor is it regular branch.*

Proposition F.55. *$\overline{\Gamma}$ is a fractal group, is weakly branch, and just-nonsolvable.*

PROOF. $\overline{\Gamma}$ is fractal by Lemma F.3 and the nature of the map ϕ. The subgroup K described above has an infinite-index derived subgroup K' (with infinite cyclic quotient), from which we conclude that $\overline{\Gamma}$ is not just-infinite; indeed K' is normal in $\overline{\Gamma}$ and $\overline{\Gamma}/K' \cong \mathbb{Z}^2 \rtimes \left(\begin{smallmatrix}-1 & 1\\ -1 & 0\end{smallmatrix}\right)$ is infinite. □

PROPOSITION F.56. *The subgroup K of $\overline{\Gamma}$ is torsion-free; thus $\overline{\Gamma}$ is virtually torsion-free.*

PROOF. For $1 \neq g \in K$, let $|g|_t$, the t-length of g, denote the minimal number of $t^{\pm 1}$'s required to write g as a word over the alphabet $\{a^{\pm 1}, t^{\pm 1}\}$. We will show by induction on $|g|_t$ that g is of infinite order.

First, if $|g|_t = 1$, i.e. $g \in \{x^{\pm 1}, y^{\pm 1}\}$, we conclude from $\phi(x^3) = (*, *, x)$ and $\phi(y^3) = (*, *, y)$ that g is of infinite order.

Suppose now that $|g|_t > 1$, and $g \in H_n \setminus H_{n+1}$. Then there is some sequence σ of length n that is fixed by g and such that $g_{|\sigma} \notin H$. By Lemma F.52, $g_{|\sigma} \in K$, so it suffices to show that all $g \in K \setminus H$ are of infinite order.

Such a g can be written as $\phi^{-1}(u, v, w)z$ for some $(u, v, w) \in \phi(K \cap H)$ and $z \in \{x^{\pm 1}, y^{\pm 1}\}$; by symmetry let us suppose $z = x$. Then $g^3 = \phi^{-1}(uavawt, vawtua, wtuava) = \phi^{-1}(g_0, g_1, g_2)$, say. For any i, we have $|g_i|_t \leq |g|_t$, because all the components of $\phi(x)$ and $\phi(y)$ have t-length ≤ 1. We distinguish three cases:

(1) $g_i = 1$ for some i. Then consider the image $\overline{g_i}$ of g_i in $\overline{\Gamma}/\overline{\Gamma}'$. By Lemma F.52, $wvu \in G'$, so $\overline{g_i} = 1 = \overline{a^2 t}$. But this is a contradiction, because $\overline{\Gamma}/\overline{\Gamma}'$ is elementary abelian of order 9, generated by the independent images $\overline{a}$ and $\overline{t}$.
(2) $0 < |g_i|_t < |g|_t$ for some i. Then by induction g_i is of infinite order, so g^3 too, and g too.
(3) $|g_i|_t = |g|_t$ for all i. We repeat the argument with g_i substituted for g. As there are finitely many elements h with $|h|_t = |g|_t$, we will eventually reach either an element of shorter length or an element already considered. In the latter case we obtain a relation of the form $\phi^n(g^{3^n}) = (\dots, g, \dots)$ from which g is seen to be of infinite order.

□

PROPOSITION F.57. *The finite quotients $\overline{\Gamma}_n = \overline{\Gamma}/H_n$ of $\overline{\Gamma}$ have order $3^{\frac{1}{4}(3^n+2n+3)}$ for $n \geq 2$, and $3^{\frac{1}{2}(3^n-1)}$ for $n \leq 2$.*

PROOF. Define the following family of two-generated finite abelian groups:

$$A_n = \begin{cases} \langle x, y \,|\, x^{3^{n/2}}, y^{3^{n/2}}, [x, y]\rangle & \text{if } n \equiv 0[2],\\ \langle x, y \,|\, x^{3^{(n+1)/2}}, y^{3^{(n+1)/2}}, (xy^{-1})^{3^{(n-1)/2}}, [x, y]\rangle & \text{if } n \equiv 1[2].\end{cases}$$

First suppose $n \geq 2$; Consider the diagram of groups described above, and quotient all the groups by H_n. Then the quotient K/K' is isomorphic to A_n, generated by x and y, and the quotient $K'/\overline{\Gamma}''$ is isomorphic to A_{n-1}, generated by $[x, y]$ and $[x, y]^t$. As $|A_n| = 3^n$, the index of K'_n in $\overline{\Gamma}_n$ is 3^{n+1} and the index of $\overline{\Gamma}''_n$ is 3^{2n}. Then as $\overline{\Gamma}''_n \cong K^3_{n-1}$ and $|\overline{\Gamma}''_2| = 1$ we deduce by induction that $|\overline{\Gamma}''_n| = 3^{\frac{1}{4}(3^n-6n+3)}$ and $|K'_n| = 3^{\frac{1}{4}(3^n-2n-1)}$, from which $|\overline{\Gamma}_n| = 3^{2n} + |\overline{\Gamma}''_n| = 3^{\frac{1}{4}(3^n+2n+3)}$ follows.

For $n \leq 2$ we have $\overline{\Gamma}_n = \mathsf{Aut}(\mathcal{T})_n = \mathbb{Z}/3 \wr \cdots \wr \mathbb{Z}/3$. □

COROLLARY F.58. *The closure $\overline{\overline{\Gamma}}$ of $\overline{\Gamma}$ in $\mathsf{Aut}(\mathcal{T})$ has Hausdorff dimension $1/2$.*

PROPOSITION F.59. *We have*

$$\phi(H) = (K' \times K' \times K') \rtimes_A \langle t_0, t_1, t_2\rangle,$$
$$\phi(K') = (K' \times K' \times K') \rtimes_B \langle d, d^t\rangle,$$

where A is such that $\langle t_0, t_1, t_2\rangle/A \cong \mathbb{Z}^4 \rtimes \mathbb{Z}/3\mathbb{Z}$ and B is such that $\langle d, d^t\rangle/B \cong \mathbb{Z}^2$.

F.7.1. The Subgroup P. Let e be the infinite sequence 2^∞, and let P be the corresponding parabolic subgroup.

THEOREM F.60. *P/P' is the direct product of $(\mathbb{Z}/3\mathbb{Z})^2$ (generated by t and $atat^{-1}a$) and an infinitely-generated free abelian group, generated by $[tt_1t, tt_2t]$. The following decomposition holds:*

$$P = \Big(K' \times K' \times \Big(\big(K' \times K' \times ((K' \times K' \times \dots) \rtimes \langle [tt_1t, tt_2t] \rangle)\big) \rtimes \langle [tt_1t, tt_2t] \rangle\Big)\Big) \rtimes \langle t, t_1t_2^{-1} \rangle,$$

where each factor (of nesting n) in the decomposition acts on the subtree just below some e_n but not containing e_{n+1}.

Define the following subgroups of Γ_n:

$$K'_n = \langle x, y \rangle'^{\Gamma_n}; \qquad Q_n = K'_n \cap P_n.$$

PROPOSITION F.61. *These subgroups have the following structure:*

$$\begin{aligned} P_n &= (K'_{n-1} \times K'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle (\mathbb{Z}^4 \ltimes \mathbb{Z}/3\mathbb{Z}) \rangle; \\ Q_n &= (K'_{n-1} \times K'_{n-1} \times Q_{n-1}) \rtimes \mathbb{Z}^2 \rangle. \end{aligned}$$

F.8. The Group $\overline{\overline{\Gamma}}$

Recall a denotes the automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let now t be the automorphism of $\mathcal{T}_3$ defined recursively by

$$t(0x\sigma) = 0\overline{x}\sigma, \qquad t(1x\sigma) = 1\overline{\overline{x}}\sigma, \qquad t(2\sigma) = 2t(\sigma).$$

Then $\overline{\overline{\Gamma}}$ is the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, t\}$; it was studied by Narain Gupta and Said Sidki [**GS83a, GS83b, Sid87a, Sid87b**].

We will use the following known facts:

THEOREM F.62. *$\overline{\overline{\Gamma}}$ is a torsion 3-group.*

PROPOSITION F.63. *We have the following diagram of normal subgroups:*

$$\begin{tikzcd}
\overline{\overline{\Gamma}} \arrow[d, dash, "\langle a \mid a^3 \rangle"] \\
H = \mathsf{Stab}_{\overline{\overline{\Gamma}}}(1) \arrow[d, dash, "\langle t \mid t^3 \rangle"] \\
\overline{\overline{\Gamma}}' = [G, H] \arrow[d, dash, "{[a,t]}"] \\
\gamma_3(\overline{\overline{\Gamma}}) = \overline{\overline{\Gamma}}^3 \arrow[d, dash, "(at)^3"] \\
H' = \phi^{-1}(\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}') = \mathsf{Stab}_{\overline{\overline{\Gamma}}}(2)
\end{tikzcd}$$

where the quotients are represented next to the arrows; all edges represent normal inclusions of index 3.

Proof. Clearly H is normal of index 3, being the kernel of the epimorphism $a \to a, t \to 1$. Then $\overline{\overline{\Gamma}}' \neq H$ (as can be checked in the finite quotient $\overline{\overline{\Gamma}}_2$) but is of index at most 3^2, so has precisely that index. Moreover, $\overline{\overline{\Gamma}}'$ is generated by the $[a^{\pm 1}, t^{\pm 1}]$: one has $[a,t]^a = [a^{-1},t][a,t]^{-1}$, $[a,t]^t = [a,t]^{-1}[a,t^{-1}]$, etc.

$\gamma_3(\overline{\overline{\Gamma}}) < \overline{\overline{\Gamma}}^3$ holds in all 3-groups, and $\overline{\overline{\Gamma}}^3$ has index 3^3 because it is 2-generated 2-step nilpotent.

Now consider H'. It is in $\mathsf{Stab}_{\overline{\overline{\Gamma}}}(2)$ since $H = \mathsf{Stab}_{\overline{\overline{\Gamma}}}(1)$. Also, $[t, t^a] = (ta)^3(a^{-1}ta^{-1})^3$ and similarly for other conjugates, so $H' < \overline{\overline{\Gamma}}^3$, and $\phi[t^{-a^2}t^{-a}, t^{-a}t^{-1}] = ([a,t],1,1)$, so $\phi(H') = \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}'$. Finally H' it is of index 3 in $\overline{\overline{\Gamma}}^3$ (since $H/H' = (\mathbb{Z}/3\mathbb{Z})^3$ on $t, t^a, t^{a^{-1}}$), and since $\mathsf{Stab}_{\overline{\overline{\Gamma}}}(2)$ is of index 3^4 in Γ (with quotient $\mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}/3\mathbb{Z}$) we have all the claimed equalities. $\square$

Proposition F.64. *$\overline{\overline{\Gamma}}$ is a just-infinite fractal group, and is a regular branch group over $\overline{\overline{\Gamma}}'$.*

Proof. $\overline{\overline{\Gamma}}$ is fractal by Lemma F.3 and the nature of the map ϕ. By direct computation, $[\overline{\overline{\Gamma}} : \overline{\overline{\Gamma}}'] = [\overline{\overline{\Gamma}}' : \phi^{-1}(\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}')] = [\phi^{-1}(\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}') : \overline{\overline{\Gamma}}''] = 3^2$, so $\overline{\overline{\Gamma}}$ is branched on $\overline{\overline{\Gamma}}'$ and is just-infinite by Proposition F.10. $\square$

Proposition F.65. *$\overline{\overline{\Gamma}}' \geq \mathsf{Stab}_{\overline{\overline{\Gamma}}}(2)$, so $\overline{\overline{\Gamma}}$ has the congruence property.*

Proposition F.66. *We have*

$$\phi(H) = (\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}') \rtimes_{3-ab} \langle t, t^a, t^{a^2} \rangle,$$
$$\phi(\overline{\overline{\Gamma}}') = (\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}') \rtimes_{3-ab} \langle [a,t], [a^2,t] \rangle.$$

F.8.1. The Subgroup P. Let e be the infinite sequence 2^∞, and let P be the corresponding parabolic subgroup.

Theorem F.67. *P/P' is an infinite elementary 3-group generated by t, $t^a t^{a^2}$ and all elements of the form $(1, \dots, 1, tt^a t^{a^2})$. The following decomposition holds:*

$$P = \left(\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \left((\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times ((\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \dots) \rtimes_{3-ab} \langle tt^a t^{a^2} \rangle)) \rtimes_{3-ab} \langle tt^a t^{a^2} \rangle\right)\right) \rtimes_{3-ab} \langle t, t^a t^{a^2} \rangle,$$

where each factor (of nesting n) in the decomposition acts on the subtree just below some e_n but not containing e_{n+1}.

Define the following subgroups of $\overline{\overline{\Gamma}}_n$:

$$\overline{\overline{\Gamma}}'_n = \langle [a,t] \rangle^{\overline{\overline{\Gamma}}_n}; \qquad Q_n = \overline{\overline{\Gamma}}'_n \cap P_n.$$

Proposition F.68. *These subgroups have the following structure:*

$$P_n = (\overline{\overline{\Gamma}}'_{n-1} \times \overline{\overline{\Gamma}}'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle t, t^a t^{a^2} \rangle;$$
$$Q_n = (\overline{\overline{\Gamma}}'_{n-1} \times \overline{\overline{\Gamma}}'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle tt^a t^{a^2} \rangle.$$

F.9. Quasi-Regular Representations

In this section we show how the information we gathered on the groups and their subgroups yields results on their representations. For G a group acting on a tree and P its parabolic subgroup, we let $\rho_{G/P}$ denote the quasi-regular representation of G on the space $\ell^2(G/P)$.

First of all consider the infinite-dimensional representations $\rho_{G/P}$. The criterion of irreducibility for quasi-regular representations was discovered by George Mackey and is as follows (the definition of commensurator is given after the theorem's statement):

THEOREM F.69 (Mackey [**Mac76, BH97**]). *Let G be an infinite group and let P be any subgroup of G. Then the quasi-regular representation $\rho_{G/P}$ is irreducible if and only if* $\mathrm{comm}_G(P) = P$.

DEFINITION F.70. *The* commensurator *of a subgroup H of G is*

$$\mathrm{comm}_G(H) = \{g \in G |\, H \cap H^g \textit{ is of finite index in } H \textit{ and } H^g\}.$$

Equivalently, letting H act on the left on the right cosets $\{gH\}$,

$$\mathrm{comm}_G(H) = \{g \in G |\, H \cdot (gH) \textit{ and } H \cdot (g^{-1}H) \textit{ are finite orbits}\}.$$

The equivalence follows, for T a finite transversal, from

$$H = \bigsqcup_{t \in T \subset H} t \cdot (H \cap H^g) \Longleftrightarrow HgH = \bigsqcup_{t \in T \subset H} t \cdot gH.$$

PROPOSITION F.71. *If G is fractal and spherically transitive, then* $\mathrm{comm}_G(P) = P$.

PROOF. Take any $g \in G$ that does not fix e; we will show that $P \cap P^g$ is of infinite index in P. Let $n \in \mathbb{N}$ be such that $f_n = g(e_n) \neq e_n$ and set $H = \mathsf{Stab}_G(n)$. Then $\phi^n(H)$ is a subgroup of G^{p^n}, and as G is spherically transitive, it projects to an infinite group in each of the factors. Similarly, $\phi(P)$ projects to an infinite group, call it Q, in all the factors except possibly that indexed by e_n, where, say, it projects to R. This R is of infinite index in Q. The projection of $\phi(P \cap P^g)$ is R in the factors indexed by e_n and f_n, so is of infinite index in $\phi(P)$. $\square$

COROLLARY F.72. *If G is fractal and spherically transitive, then $\rho_{G/P}$ is irreducible.*

The quasi-regular representations we consider are good approximants of the regular representation in the following sense:

THEOREM F.73. *ρ_G is a subrepresentation of $\bigotimes_{P \text{ parabolic}} \rho_{G/P}$.*

PROOF. Since $\bigcap_{g \in G} P^g = 1$, it follows that the G-space G is a subspace of $\prod_{g \in G} G/P_g$. The representation on a product of spaces is the tensor product of the representation on the spaces. $\square$

We have a continuum of parabolic subgroups $P_e = \mathsf{Stab}_G(e)$, where e runs through the boundary of a tree, so formally we also have a continuum of quasi-regular representations. If G is countable, there are uncountably many non-equivalent representations, because among the uncountably many P_e only countably many are conjugate. We therefore have the

THEOREM F.74. *There are uncountably many non-equivalent representations of the form $\rho_{G/P}$, where P is a parabolic subgroup.*

We now consider the finite-dimensional representations ρ_{G/P_n}, where P_n is the stabilizer of the vertex at level n in the ray defining P. These are permutational representations on the sets G/P_n. The ρ_{G/P_n} are factors of the representation $\rho_{G/P}$. Noting that $P = \bigcap_{n \geq 0} P_n$, it follows that

$$\rho_{G/P_n} \Rightarrow \rho_{G/P},$$

in the sense that for any non-trivial $g \in G$ there is an $n \in \mathbb{N}$ with $\rho_{G/P_n}(g) \neq 1$.

F.9.1. Hecke Algebras. Corollary F.72 showed that the quasi-regular representation $\rho_{G/P}$ is irreducible for all of our examples. We now describe the decomposition of the finite quasi-regular representations ρ_{G/P_n}. It turns out that it is closely related to the orbit structure of P_n on G/P_n, through the *Hecke algebra*. The result we shall prove is:

THEOREM F.75. *ρ_{G/P_n} and $\rho_{\tilde{G}/\tilde{P}_n}$ decompose as a direct sum of $n+1$ irreducible components, one of degree 2^i for each $i \in \{1, \dots, n-1\}$ and two of degree 1.*

$\rho_{\Gamma/P}$, $\rho_{\overline{\Gamma}/P}$ and $\rho_{\overline{\overline{\Gamma}}/P}$ decompose as a direct sum of $2n+1$ irreducible components, two of degree 2^i for each $i \in \{1, \dots, n-1\}$ and three of degree 1.

The proof of this theorem will appear after the following definitions and lemmata.

DEFINITION F.76. *Let G be a group and P a subgroup. The* Hecke algebra *(also called the* intersection algebra*) $\mathcal{H}(G,P)$ is the algebra* $\mathrm{End}_{\ell^2(G)}(\ell^2(G/P)$*. It can be seen as the algebra of (P,P)-biinvariant functions on G, with the convolution product*

$$(f \cdot g)(x) = \sum_{y \in G} f(xy)g(y^{-1}).$$

$\mathcal{H}(G,P)$ is spanned by the $(P-P)$-biinvariant functions on G, or equivalently by the double cosets PgP. The following result stresses the importance of the Hecke algebra in the study of representation decomposition:

THEOREM F.77 ([**CR90**, Section 11D]). *$\mathcal{H}(G,P)$ is a semi-simple algebra. There is a canonical bijection between irreducible components of $\rho_{G/P}$ and simple factors of $\mathcal{H}(G,P)$, occurring with the same multiplicities.*

Then, if $\mathcal{H}(G,P)$ is abelian, its decomposition in simple modules is has as many components as there are double cosets PgP in G.

In our examples, the spaces have the following order of magnitude: the core of P_n is the normal subgroup $H_n = \bigcap_{g \in G} P_n^g$, of index $\sim e^{e^n}$. The subgroup P_n is of index $\sim e^n$. The number of double cosets is $\sim n$. We give the precise results for our five examples.

F.9.2. Orbits In G/P_n. As the double cosets $P_n g P_n$ are in one-to-one correspondence with the orbits of P_n on G/P_n we shall now describe the orbits for this action.

LEMMA F.78. *There are two K_n-orbits on Σ^n: those sequences starting with 0 and those starting with 1.*

P_n has $n+1$ orbits in Σ^n; they are 1^n and the $1^i0\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $1^i0\Sigma^*$ for all $i \in \mathbb{N}$.

PROOF. As K_n contains $K_{n-1} \times K_{n-1}$, it follows by induction that K_n acts transitively on the sets $00\Sigma^{n-2}$ and $01\Sigma^{n-2}$. As K_n contains $(ab)^2 = (ca, ac)$, it also permutes $00\Sigma^{n-2}$ and $01\Sigma^{n-2}$, so it acts transitively on $0\Sigma^{n-1}$. The same holds for $1\Sigma^{n-1}$.

The last assertion follows from Theorem F.25. □

LEMMA F.79. *There are two $\tilde{K}_n$-orbits on Σ^n: those sequences starting with 0 and those starting with 1.*

$\tilde{P}_n$ has $n+1$ orbits in Σ^n; they are 1^n and the $1^i0\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of $\tilde{P}$ in $\mathcal{T}_\Sigma$ are the $1^i0\Sigma^*$ for all $i \in \mathbb{N}$.

PROOF. Completely similar to F.78. □

LEMMA F.80. *There are three Γ'_n-orbits on Σ^n: those sequences starting with 0, those starting with 1 and those starting with 2.*

P_n has $2n+1$ orbits in Σ^n; they are 2^n and the $2^i0\Sigma^{n-1-i}$ and $2^i1\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $2^i0\Sigma^*$ and $2^i1\Sigma^*$ for all $i \in \mathbb{N}$.

PROOF. As Γ'_n contains $\Gamma'_{n-1} \times \Gamma'_{n-1} \times \Gamma'_{n-1}$, it follows by induction that Γ'_n acts transitively on the sets $00\Sigma^{n-2}$, $01\Sigma^{n-2}$ and $02\Sigma^{n-2}$. As Γ'_n contains $[a,t] = (ta^{-1}, a, t^{-1})$, it also permutes $00\Sigma^{n-2}$, $01\Sigma^{n-2}$ and $02\Sigma^{n-2}$, so it acts transitively on $0\Sigma^{n-1}$. The same holds for $1\Sigma^{n-1}$ and $2\Sigma^{n-1}$.

The last assertion follows from Theorem F.50 □

LEMMA F.81. *For the group $\overline{\Gamma}$, there are three K'_n-orbits on Σ^n: those sequences starting with 0, those starting with 1 and those starting with 2.*

P_n has $2n+1$ orbits in Σ^n; they are 2^n and the $2^i0\Sigma^{n-1-i}$ and $2^i1\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $2^i0\Sigma^*$ and $2^i1\Sigma^*$ for all $i \in \mathbb{N}$.

Proof. As K'_n contains $K'_{n-1} \times K'_{n-1} \times K'_{n-1}$, it follows by induction that K'_n acts transitively on the sets $00\Sigma^{n-2}$, $01\Sigma^{n-2}$ and $02\Sigma^{n-2}$. As K'_n contains $[x, y] = (at, at, ta)$, it also permutes $00\Sigma^{n-2}$, $01\Sigma^{n-2}$ and $02\Sigma^{n-2}$, so it acts transitively on $0\Sigma^{n-1}$. The same holds for $1\Sigma^{n-1}$ and $2\Sigma^{n-1}$.

The last assertion follows from Theorem F.60 $\square$

Lemma F.82. *There are three $\overline{\overline{\Gamma}}'_n$-orbits on Σ^n: those sequences starting with* 0, *those starting with* 1 *and those starting with* 2.

P_n has $2n + 1$ orbits in Σ^n; they are 2^n and the $2^i0\Sigma^{n-1-i}$ and $2^i1\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $2^i0\Sigma^*$ and $2^i1\Sigma^*$ for all $i \in \mathbb{N}$.

Proof. Completely similar to F.80. $\square$

F.9.3. Gelfand Pairs. We have seen the Hecke algebra $\mathcal{H}(G, P_n)$ is roughly of dimension n. Its structure is further simplified by the following consideration:

Definition F.83 ([**Dia88**]). *Let G be a group and P any subgroup. The pair (G, P) is a* Gelfand pair *if all irreducible subrepresentations of $\rho_{G/P}$ have multiplicity* 1.

Lemma F.84 ([**CR90**, Exercise 18, page 306],[**Mac76**, Theorem 1.20]). *(G, P) is a Gelfand pair if and only if $\mathcal{H}(G, P)$ is abelian.*

Proposition F.85. *In our five examples the pairs (G, P_n) form a Gelfand pair for all $n \in \mathbb{N}$.*

Proof. Clearly $P_0 = G$ so $\mathcal{H}(G, P_0) = \mathbb{C}$ is abelian. Furthermore, P_{n+1} is a subgroup of P_n, and the natural map $G/H_{n+1} \twoheadrightarrow G/H_n$ induces a map $P_{n+1}/H_{n+1} \twoheadrightarrow P_n/H_n$, so $\mathcal{H}(G, P_n) \cong \mathcal{H}(G/H_n, P_n/H_n)$ is a direct summand of $\mathcal{H}(G, P_{n+1})$, and their dimensions differ by $d - 1$, which is 1 or 2 (recall d is the degree of the regular tree on which G acts). Now writing

$$\mathcal{H}(G, P_{n+1}) = \mathcal{H}(G, P_n) \oplus A,$$

we see that A is semi-simple and of dimension $d - 1 < 4$. All such semisimple algebras are abelian, $A \cong \mathbb{C}^{d-1}$, so $\mathcal{H}(G, P_{n+1})$ is abelian too. $\square$

Proof of Theorem F.75. By Proposition F.85, the Hecke algebra $\mathcal{H}(G, P_n)$ is abelian, so it is isomorphic to $\mathbb{C}^{N_n}$, where N_n is its dimension. This N_n in turn is equal to the number of double cosets $P_n g P_n$. These numbers N_n are computed in the corollaries in Subsection F.9.2. By Theorem F.77, the number of irreducible subrepresentations of ρ_{G/P_n} is N_n. Finally, $\rho_{G/P_n} = \rho_{G/P_{n-1}} \oplus A_{n,1} \oplus \cdots \oplus A_{n,d-1}$, where the $A_{n,i}$ are irreducible representations. Since $\dim \rho_{G/P_n} = d^n$ and $\dim A_i$ is a power of d, the only possibility is that $\dim A_{n,i} = d^{n-1}$ for all $i \in \{1, \dots, d-1\}$, and

$$\rho_{G/P_n} = \rho_{G/P_0} \oplus A_{1,1} \oplus \cdots \oplus A_{1,d-1} \oplus \cdots \oplus A_{n,1} \oplus \cdots \oplus A_{n,d-1}.$$

$\square$

It may well be that for all *GGS* groups the Hecke algebra associated to a parabolic subgroup is commutative.

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